MATH20411 **Problem sheet 6: Separation of Variables B**

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1. Find all the separated solutions for the deflections $u(x,t)$ of a stretched string satisfying the PDE $u_{tt} = u_{xx}$ and the boundary conditions:

$$a)\quad u(0,t)=0, \quad u(L,t)=0, \qquad b)\quad u_x(0,t)=0, \quad u(L,t)=0,$$

where L is the length of the string. Given initial conditions,

$$u(x,0)=f(x), \quad u_t(x,0)=g(x),$$

derive an infinite series expression for the solution in each case.

2. Use the method of separation of variables to show that the solution $u(x,t)$ to Laplace's equation: $u_{xx}+u_{yy}=0$, on the rectangular domain with $x \in [0,\pi]$ and $y \in [0,1]$, that also satisfies the boundary conditions:

$$u(0,y)=u(\pi,y)=0 \quad \text{for} \quad y \in [0,1]$$
$$u(x,0)=0 \quad \text{and} \quad u(x,1)=\pi \quad \text{for} \quad x \in [0,\pi].$$

can be written as the infinite series:

$$u = \sum_{k=0}^{\infty} \frac{4}{2k+1} \frac{\sinh\big((2k+1)y\big)}{\sinh(2k+1)} \sin\big((2k+1)x\big).$$

3. Recall that the wave equation (with wave speed $c^2=1$) is

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u.$$

We can use it to model the vibrations of the membrane on top of a circular drum of radius a. u denotes the deflection of the circular membrane from the (x,y) plane.

a) Write down the wave equation in polar co-ordinates and show that if the solution u does not depend on the angle θ then $u=u(r,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}. \tag{1}$$

b) Show that the separated solutions to (1) for $0<r<a$ and $t>0$, subject to $|u(0,t)|<\infty$ and $u(a,t)=0$, are

$$u_n(r,t) = \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) J_0(\omega_n r), \qquad n=1,2,3\ldots$$

where $\omega_n = \frac{\alpha_n}{a}$ and α_n is the nth root of the Bessel function J_0 (see handout). You only need to consider negative values of the separation constant.

c) Let $a=1$ and find a solution that also satisfies the initial conditions $u(r,0) = J_0(\alpha_3 r)$, $\frac{\partial u}{\partial t}(r,0)=0$.