MATH36001 Mid-Term Test: Information 2015

The mid-term test will take place during the Wednesday lecture on **on 11th November.** The rooms are **Stopford Lecture Theatre 2 and 6**. (Refer to the course webpage for info!)
The rubric is to answer all the questions, which are worth a total of 20 marks.
The test is on all the material discussed in the lectures and tutorials up to date.
Below is a sample test.

Total 20 marks. Answer all 8 questions.

1. True or false: $p(A)A = Ap(A)$ for any $A \in \mathbb{C}^{n\times n}$ and any polynomial p? Justify your answer with a proof or a counterexample. [3 marks]

♠ True — if $p(z) = c_0 + c_1 z + \cdots + c_k z^k$, then $p(A) = c_0 I + c_1 A + \cdots + c_k A^k$. Now, $A^j = A^{j-1}A = AA^{j-1}$ so that $Ap(A) = c_0 AI + c_1 AA + \cdots + c_k AA^k = c_0 IA + c_1 AA + \cdots + c_k A^k A = p(A)A$.

2. For $A \in \mathbb{C}^{n\times n}$, give two conditions equivalent to A being singular. [2 marks]

♠ See Theorem 1 in the *Basic material* handout.

3. For nonzero $u, v \in \mathbb{C}^n$ let $A = I - uv^*$. Show that $\mu = 1 - v^*u$ is an eigenvalue of A with associated eigenvector u. What are the other eigenvalues of A? For which value of μ is A is idempotent (i.e., $A^2 = A$)? [4 marks]

♠ Note that $(I - uv^*)u = (1 - v^*u)u$ so $\mu = 1 - v^*u$ is an eigenvalue of A. Let $v_2, \ldots, v_n$ be $n-1$ linearly independent vectors such that $v^*v_i = 0$ for $i = 2, \ldots, n$. Then $(I - uv^*)v_i = v_i$ so $\lambda = 1$ is an eigenvalue of multiplicity $n-1$.

Since $A^2 = (I - uv^*)(I - uv^*) = I - (2 - v^*u)uv^*$ and u, v are nonzero, $A^2 = A$ if and only $2 - v^*u = 1$, that is, $\mu = 0$.

4. State the spectral theorem. [2 marks]

5. True or false: *An $n \times n$ matrix has a set of n linearly independent eigenvectors.* Justify your answer with a proof or a counterexample.

[1 marks]

♠ False — for example the 2×2 Jordan block $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has only one eigenvector.

6. Ascertain if the following matrices are unitarily similar to a diagonal matrix:

- A such that $A^*A = AA^*$.
- B has n orthogonal eigenvectors.
- C has n distinct eigenvalues.

[3 marks]

♠ A and B are both normal so they are unitarily similar to a diagonal matrix. The matrix C is diagonalizable but not necessarily by a unitary transformation.

7. Suppose the Jordan form of A is

$$J = \operatorname{diag}\left(\begin{bmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ & 3 \end{bmatrix}, [\,2\,], [\,2\,] \right).$$

(a) Determine the algebraic and geometric multiplicity of each distinct eigenvalue of A.

(b) Give the linearly independent eigenvectors of the Jordan matrix J.

(c) Determine the minimal polynomial of A.

[3 marks]

♠ Incomplete solution: the eigenvalue $\lambda = 4$ has algebraic multiplicity 5 and geometric multiplicity 2. It has two linearly independent eigenvectors, for example e_1 and e_4. Here e_i is the ith column of the 9×9 identity matrix.

The minimal polynomial is $q(x) = (x-4)^3(x-3)^2(x-2)$.

8. Define the vector p-norm and state the Hölder inequality. [2 marks]

♠ See handout on Norms, page 1.