# MATH36001 Solutions: SVD 2015

**1**. We have that $z^*A^*Az = \|Az\|_2^2 \geq 0 \ \forall \ z \neq 0$. Since $\mathrm{rank}(A) = n$, $A$ has full rank, that is, $Az = 0$ iff $z = 0$. Hence $z^*A^*Az > 0$ for all nonzero $z$ which shows that $A^*A$ is Hermitian positive definite and therefore nonsingular.

Let $X = (A^*A)^{-1}A^*$. Then it is easy to check that $X$ satisfies the four Moore–Penrose conditions.

**2**.

For any $A \in \mathbb{C}^{m\times n}$, $\mathrm{rank}(A) + \dim(\mathrm{null}(A)) = n$. Hence $\dim(\mathrm{null}(A)) = n - r$ and $\dim(\mathrm{range}(A)) = \mathrm{rank}(A) = r$.

From $AV = U\Sigma$ we have

$$\begin{aligned} Av_i &= \sigma_i u_i,\ (\sigma_i \neq 0) \quad i = 1,\dots,r,\\ Av_i &= 0, \quad i = r+1,\dots,n. \end{aligned}$$

$U$ unitary $\Rightarrow$ the vectors $u_i$, $i = 1,\dots,r$, are linearly independent and therefore $\mathrm{range}(A) = \mathrm{span}\{u_1, u_2, \dots, u_r\}$.
$V$ unitary $\Rightarrow$ the vectors $v_i$, $i = r+1,\dots,n$, are linearly independent and therefore $\mathrm{null}(A) = \mathrm{span}\{v_{r+1},\dots,v_n\}$.

**3**. Let $A$ have the SVD $A = U\Sigma V^*$. By the unitary invariance of the 2 and Frobenius norms,

$$\begin{aligned} \|A\|_2 &= \|\Sigma\|_2 = \sigma_1,\\ \|A\|_F &= \|\Sigma\|_F = \Big(\sum_{i=1}^{n} \sigma_i^2\Big)^{1/2}. \end{aligned}$$

Thus

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$$

(in fact, we can replace $\sqrt{n}$ by $\sqrt{r}$, where $r = \mathrm{rank}(A)$).
There is equality on the left when $\sigma_2 = \cdots = \sigma_n = 0$, that is, when $A$ has rank 1 ($A = xy^*$) or $A = O$. There is equality on the right when $\sigma_1 = \cdots = \sigma_n = \alpha$, that is, when $A = \alpha Q$ where $Q$ is unitary, $\alpha \in \mathbb{C}$.

**4**. The easiest is to notice that

$$\|AX - I_m\|_F^2 = \|A[x_1,\dots,x_m] - [e_1,\dots,e_m]\|_F^2 = \sum_{i=1}^{m} \|Ax_i - e_i\|_2^2,$$

which is minimized if $\|Ax_i - e_i\|_2$ is minimized for $i = 1,\dots,m$. Thus we have $m$ simultaneous, but independent, LS problems. The solution is $x_i = A^+e_i$, $i = 1,\dots,m$, that is, $X = A^+I_m = A^+$. This is the unique solution only if $A$ has full rank, but it is always the unique *minimum F-norm* solution. An alternative proof is to consider $f(\alpha) = \|A(X + \alpha Y) - I_m\|_F^2$ and derive conditions for $f(0) = \min_\alpha f(\alpha)$. Another (longer) proof uses the SVD of $A$ and works from first principles.

**5**. (i)

$$\begin{aligned}\|A-QB\|_F^2 &= \operatorname{trace}((A-QB)^T(A-QB))\\ &= \operatorname{trace}(A^TA)+\operatorname{trace}(B^TQ^TQB)-\operatorname{trace}(B^TQ^TA)-\operatorname{trace}(A^TQB)\\ &= \text{const}-2\operatorname{trace}(A^TQB),\end{aligned}$$

where const denotes a term independent of $Q$. Hence minimizing $\|A-QB\|_F$ is equivalent to maximizing $\operatorname{trace}(A^TQB)$.

(ii) We have

$$\operatorname{trace}(A^TQB)=\operatorname{trace}(QBA^T)=\operatorname{trace}(QU\Sigma V^T)=\operatorname{trace}(V^TQU\Sigma)=\operatorname{trace}(Z\Sigma).$$

Since $Z$ is orthogonal, all its elements lie between $-1$ and 1, hence $\operatorname{trace}(Z\Sigma)=\sum_i z_{ii}\sigma_i \le \sum_{i=1}^m \sigma_i$.

(iii) Equality holds for $Z=I$, that is, $Q=VU^T$.

(iv) Theorem: Let $A,B\in\mathbb{R}^{m\times n}$. Then $\|A-QB\|_F$ with $Q$ orthogonal is minimized by $Q=VU^T$, where $BA^T=U\Sigma V^T$ is an SVD.

**6**. Let $P_1$ and $P_2$ be orthogonal projectors onto $S$. Since $\operatorname{range}(P_1)=\operatorname{range}(P_2)$, $P_2=P_1X$ for some $X$. Then $P_1P_2=P_1^2X=P_1X=P_2$. Likewise, $P_2P_1=P_1$. Hence, for any $z$,

$$\begin{aligned}\|(P_1-P_2)z\|_2^2 &= z^*(P_1-P_2)(P_1-P_2)z\\ &= z^*(P_1^2+P_2^2-P_1P_2-P_2P_1)z\\ &= z^*(P_1+P_2-P_2-P_1)z=0.\end{aligned}$$

Therefore $P_1-P_2=O$.

**7**.

(i) Let $A=U\Sigma V^*$ be the singular value decomposition of $A$, $r=\operatorname{rank}(A)$ and $U=[u_1,\dots,u_m]$. Then in a similar way to Exercise **??** we find that $\operatorname{null}(A^*)=\operatorname{span}\{u_{r+1},\dots,u_m\}$. But by (**??**), $\operatorname{range}(A)=\operatorname{span}\{u_1,\dots,u_r\}$ so $\big(\operatorname{range}(A)\big)^\perp=\operatorname{span}\{u_{r+1},\dots,u_m\}$ and hence, $\big(\operatorname{range}(A)\big)^\perp=\operatorname{null}(A^*)$. It follows that $\big(\operatorname{range}(A^*)\big)^\perp=\operatorname{null}(A)$ and hence $\operatorname{range}(A^*)=\big(\operatorname{null}(A)\big)^\perp$.

(ii) $AA^+$ is Hermitian by condition (iii) of the Moore–Penrose conditions, and $(AA^+)^2=AA^+AA^+=AA^+$ by condition (i) of the Moore–Penrose conditions. It remains to show that $\operatorname{range}(AA^+)=\operatorname{range}(A)$.

Let $x\in\operatorname{range}(A)$, so that $x=Ay$ for some $y$. Then, by condition (i) of the Moore–Penrose conditions, $x=AA^+Ay=AA^+x$, so $x\in\operatorname{range}(AA^+)$. Conversely, $x\in\operatorname{range}(AA^+)$ implies $x=AA^+y$ for some $y$ and then $x=A(A^+y)\in\operatorname{range}(A)$. Thus $P_{\operatorname{range}(A)}=AA^+$ is the orthogonal projector onto $\operatorname{range}(A)$. Since $\big(\operatorname{range}(A)\big)^\perp=\operatorname{null}(A^*)$, $P_{\operatorname{null}(A^*)}=I-AA^+$ is the orthogonal projector onto $\operatorname{null}(A^*)$.

By the first part, the orthogonal projector onto $\operatorname{range}(A^*)$ is $A^*(A^*)^+=A^*(A^+)^*=(A^+A)^*=A^+A$. Since $\big(\operatorname{range}(A^*)\big)^\perp=\operatorname{null}(A)$, $P_{\operatorname{null}(A)}=I-A^+A$ is the orthogonal projector onto $\operatorname{null}(A)$.

(iii) Computing the products $AA^+$ and $A^+A$ reveals that

$$AA^+ = U\Sigma V^* V\Sigma^+ U^* = U \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} U^* = U_1 U_1^*, \quad A^+A = V\Sigma^+ U^* U\Sigma V^* = V \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} V^* = V_1 V_1^*$$

and since $U$ and $V$ are unitary, $U_2U_2^* = I - U_1U_1^*$ and $V_2V_2^* = I - V_1V_1^*$. The result follows from (ii).

**8**. It is easy to verify that $x^+ = x^*/(x^*x)$ satisfies the four Moore–Penrose conditions.

**9**. Yes. This follows from the symmetry in $A$ and $X$ of the Moore–Penrose conditions (i.e., the roles of $A$ and $X$ can be interchanged).