MATH36001 Solutions: Perron–Frobenius 2015

1.

(a) $(Ax)_i = \sum_{j=1}^{n} a_{ij}x_j > 0$ since $a_{ij} > 0$ and not all x_j are zero.

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) If $z = w$ then clearly $Az = Aw$. Now assume that $z \neq w$ so that $z - w \geq 0$, $z - w \neq 0$ and by (a), $A(z - w) > 0$, that is, $Az > Aw$.

2. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $a_{ij} > 0$ for all i, j. If A is stochastic then $a_{11} + a_{12} = 1$ and $a_{21} + a_{22} = 1$ which which together with $a_{ij} > 0$ implies that $a_{ij} < 1$ for all i, j. If we denote $a_{12} = \alpha$ and $a_{21} = \beta$ then $A = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$ with $0 < \alpha, \beta < 1$.

Since A is positive stochastic, $\rho(A) = \rho(A^T) = 1$ is the Perron root. The Perron vector $p > 0$ solves $(I - A^T)p = 0$, $\|p\|_1 = 1$, that is, $\begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with $p_1 + p_2 = 1$. This yields $p = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.

3. The graph of A is strongly connected so A is irreducible. Note that $A^T e = 3e$, where $e = [1 \quad 1 \quad 1]^T$ so $\rho(A) \geq 3$. But $\rho(A) \leq \|A\|_1 = 3$ so the Perron root $\rho(A) = 3$. An easy calculation shows that the Perron vector is $p = [1/6 \quad 1/2 \quad 1/3]$.

4. If (λ, y) is an eigenpair of A such that $y \geq 0$ and if $x > 0$ is the Perron vector of A^T then $x^T y > 0$ and $\rho(A)x^T = x^T A \;\Rightarrow\; \rho(A)x^T y = x^T Ay = \lambda x^T y \;\Rightarrow\; \rho(A) = \lambda$. Since $\rho(A)$ is a simple eigenvalue (Perron–Frobenius theorem (iv)), y must be a positive multiple of x.

5. If P is irreducible and stochastic then $\rho(P) = 1$ is a simple eigenvalue for P. Consequently,

$$\operatorname{rank}(I - P) = n - \dim(\operatorname{null}(I - P)) = n - \text{geom mult of } 1 = n - \text{alg mult of } 1 = n - 1.$$

6. By definition $p_{ij} = e_i^T P e_j = \rho(A)^{-1} x_i^{-1} a_{ij} x_j$ and since $Ax = \rho(A)x$,

$$\sum_{j=1}^{n} a_{ij}x_j = \rho(A)x_i, \qquad i = 1, 2, \ldots, n.$$

Hence

$$\sum_{j=1}^{n} p_{ij} = \rho(A)^{-1} x_i^{-1} \sum_{j=1}^{n} a_{ij}x_j = \rho(A)^{-1} x_i^{-1} \rho(A) x_i = 1.$$

Thus P is stochastic.

This exercise shows that a wide class of nonnegative matrices can be reduced to stochastic matrices by simple diagonal scaling.