MATH36001 Solutions: Background Material 2015

1. (i)

$$b = \left[\begin{array}{cc} 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \end{array}\right] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} [1 \quad 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ [3 \quad 4] \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ [5 \quad 6] \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1+4 \\ 3+8 \\ 5+12 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}.$$

(ii)

$$\left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}\right] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}.$$

2. $A = \sum_{j=1}^{n-1} e_j e_{j+1}^T$.

3. (i)

$$AA^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} + \frac{1}{4} & 0 & -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} & 0 & \frac{1}{4} - \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Since $AA^T = I$ we have $A^{-1} = A^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(ii) Let Q_1, Q_2 be orthogonal. Then $(Q_1Q_2)(Q_1Q_2)^T = Q_1Q_2Q_2^TQ_1^T = Q_1Q_1^T = I$.

(iii) $1 = \det I = \det(Q^TQ) = \det Q^T \det Q = (\det Q)^2$ so $\det(Q) = \pm 1$.

$1 = \det I = \det(U^*U) = \det U^* \det U = \overline{\det U} \det U$ so $\det U = e^{i\theta}$.

4. (i) $X = D - CA^{-1}B$ and $Y = A - BD^{-1}C$.
(ii) From the block LU factorization of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ we deduce that

$$\det\left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\right) = \det\left(\left[\begin{smallmatrix} I & O \\ CA^{-1} & I \end{smallmatrix}\right]\right) \det\left(\left[\begin{smallmatrix} A & B \\ O & D-CA^{-1}B \end{smallmatrix}\right]\right) = \det(A)\det(D - CA^{-1}B),$$

where we have used the fact that the determinant of a block-triangular matrix is the product of determinants of the diagonal blocks, and that $\det(I) = 1$. Likewise, the block UL factorization of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ leads to $\det\left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\right) = \det(D)\det(A - BD^{-1}C)$.
Setting $A = I$ and $D = I$ in (1) yields (2). With $B = x$ and $C = y^*$ so that $n = 1$, (2) yields $\det(I - xy^*) = \det(1 - y^*x) = 1 - y^*x$.

5. (i)

$$\begin{aligned}(A+uv^*)\big(A^{-1}-\frac{A^{-1}uv^*A^{-1}}{1+v^*A^{-1}u}\big) &= I-\frac{uv^*A^{-1}}{1+v^*A^{-1}u}+uv^*A^{-1}-\frac{uv^*A^{-1}uv^*A^{-1}}{1+v^*A^{-1}u}\\ &= I-\frac{uv^*A^{-1}}{1+v^*A^{-1}u}(1+v^*A^{-1}u)+uv^*A^{-1}\\ &= I.\end{aligned}$$

(ii) From Exercise (4), $\det(I+\alpha e_i e_j^T) = 1+\alpha e_j^T e_i$. Thus for $i \neq j$, $I+\alpha e_i e_j^T$ is always nonsingular. When $i=j$, $I+\alpha e_i e_i^T$ is nonsingular as long as $\alpha \neq -1$. In this case, the Sherman–Morrison formula gives $(I+\alpha e_i e_j^T)^{-1} = I-\alpha(1+\alpha e_j^T e_i)^{-1} e_i e_j^T$.

6. Suppose $x_1,\dots,x_n$ are linearly dependent so that $\sum_{i=1}^n \alpha_i x_i = 0$ with not all constants $\alpha_1,\dots,\alpha_n$ equal to zero. Then for $j=1,\dots,n$, $x_j^*\sum_{i=1}^n \alpha_i x_i = \alpha_j x_j^* x_j = 0$ which implies that $\alpha_j = 0$ since $x_j^* x_j \neq 0$.

7.

(i) Identity matrix.

(ii) Zero matrix.

(iii) xy^* idempotent when $y^*x=1$ and nilpotent when $y^*x=0$.

(iv) $AB = AABB = ABAB = (AB)^2$.

(v) Suppose $A^2 = A$ and A nonsingular. Then $A^{-1}(A^2) = A^{-1}A \Leftrightarrow A = I$.

(vi) $A^2 - I = (A-I)(A+I)$. So $A^2 = I$ if and only if $(A-I)(A+I) = (A+I)(A-I) = O$.

(vii) $(I-2xx^*)(I-2xx^*) = I - 4xx^* + 4xx^* = I$.

(viii) $I = (I-A)(I-A)^{-1} = (I-A)\sum_{j=0}^k A^j = \sum_{j=0}^k A^j - \sum_{j=1}^{k+1} A^j = I - A^{k+1} \Rightarrow A^{k+1} = O$.

8. (i) If $T = AB$ is a product of upper triangular matrices, then, accounting for the zero subdiagonals of A and B,

$$t_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=i}^j a_{ik}b_{kj}, \qquad (1)$$

and the product is zero if $j < i$, as required. Transposing the equation $T = AB$ gives $T^T = B^T A^T$, which implies the corresponding result for lower triangular matrices.

Setting $i = j$ in (1) gives that $t_{ii} = e_i^T ABe_i = a_{ii}b_{ii}$.

(ii) $\det(A) = \prod_{j=1}^n a_{jj} \neq 0$ iff $a_{jj} \neq 0$ for all j. Regarding the inverse, let $T = A^{-1}$. Since $TA = I$ we have, equating $(i,1)$ elements,

$$0 = \sum_{k=1}^n t_{ik}a_{k1} = t_{i1}a_{11}, \qquad i > 1.$$

Since $a_{11} \neq 0$ this implies $t_{21} = \cdots = t_{n1} = 0$. For $i > j = 2$, we then have $0 = t_{i2}a_{22}$, showing that T is zero beneath the diagonal in the second column. 'By induction' T is upper triangular.

Using (i) with $B = A^{-1}$ we have that $1 = e_j^T I e_j = e_j^T A B e_j = a_{jj}b_{jj} \;\Rightarrow\; b_{jj} = 1/a_{jj}$.

9.

(i) $\operatorname{trace}(\alpha A + \beta B) = \sum \alpha a_{ii} + \sum \beta b_{ii} = \alpha \sum a_{ii} + \beta \sum b_{ii} = \alpha \operatorname{trace}(A) + \beta \operatorname{trace}(B)$.

(ii) Since $(AB)_{ii} = \sum_{k=1}^n a_{ik}b_{ki}$,

$$\operatorname{trace}(AB) = \sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki} = \sum_{i=1}^n \sum_{k=1}^n b_{ik}a_{ki} = \operatorname{trace}(BA).$$

(iii) $0 = S + S^T \Rightarrow \operatorname{trace}(S + S^T) = 2\operatorname{trace}(S) = 0 \Rightarrow \operatorname{trace}(S) = 0$.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $\operatorname{trace}(A) = 0$ but A is not skew-symmetric.