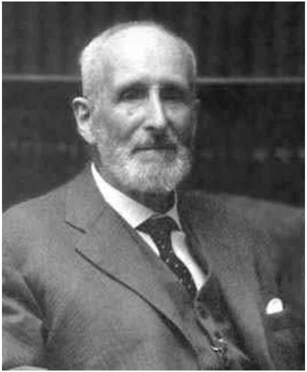
Perron–Frobenius Theory



Oskar Perron
(1880–1975)

Georg Frobenius
(1849–1917)

Positive and Nonnegative Matrices

Let $A, B \in \mathbb{R}^{m \times n}$.

- $A \geq B$ if $a_{ij} \geq b_{ij} \; \forall \; i, j$,
- $A > B$ if $a_{ij} > b_{ij} \; \forall \; i, j$,
- A is **nonnegative** if $A \geq 0$,
- A is **positive** if $A > 0$.

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What about the eigenvalues of $A \geq 0$?

Spectral radius: $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

Theorem (Nonnegative eigenpairs, Thm. 1)

If $A \geq 0$ then $\rho(A)$ is an eigenvalue of A and there exists an associated eigenvector $x \geq 0$ such that $Ax = \rho(A)x$.

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The following lemma is a consequence of Theorem 1.

Lemma (Lem. 2)

Let $A \geq 0$. Then $I - A$ is nonsingular and $(I - A)^{-1} \geq 0$ if and only if $\rho(A) < 1$.

Positive Matrices

Theorem (Perron's theorem, Thm. 3)

If $A \in \mathbb{R}^{n\times n}$ and $A > 0$ then

(i) $\rho(A) > 0$.

(ii) $\rho(A)$ *is an e'val of A.*

(iii) *There is an e'vec x with* $x > 0$ *and* $Ax = \rho(A)\,x$.

(iv) *The e'val* $\rho(A)$ *has algebraic multiplicity 1.*

(v) *All the other e'vals are less than* $\rho(A)$ *in absolute value, i.e.,* $\rho(A)$ *is the only e'val of maximum modulus.*

(Proof not examinable.)

Powers of positive matrices

Theorem (Thm. 4)

If $A > 0$, x is any positive e'vec of A corresponding to $\rho(A)$, and y is any positive e'vec of A^T corresponding to $\rho(A) = \rho(A^T)$ then

$$\lim_{k\to\infty} \left(\frac{A}{\rho(A)}\right)^k = \frac{xy^T}{y^T x} > 0.$$

Irreducible Nonnegative Matrices

Most properties in Perron's Thm are lost for $A \geq 0$ unless $A \in \mathbb{R}^{n \times n}$ is **irreducible** (i.e., not reducible).

$A \in \mathbb{R}^{n \times n}$ is **reducible** if there exists a permutation matrix P s.t.

$$P^T A P = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix},$$

where X and Z are both square.

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Directed graph of $A \in \mathbb{R}^{n\times n}$: connects n pts $P_1, \ldots, P_n$ by a direct link from P_i to P_j if $a_{ij} \neq 0$.

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Fact

$A \geq 0$ is irreducible if and only if its directed graph is strongly connected.

Perron–Frobenius theorem

Theorem (Thm.5)

If $A \geq 0$ is irreducible then

(i) $\rho(A) > 0$.

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(iii) *There is an e'vec* x *with* $x > 0$ *and* $Ax = \rho(A)\,x$.

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$\lambda_{\max}(A) = \rho(A)$ is called the **Perron root**.

The **Perron vector** is the unique vector p defined by

$$Ap = \rho(A)p, \quad p > 0, \quad \|p\|_1 = 1.$$

Stochastic Matrices

$P \in \mathbb{R}^{n \times n}$ is a **stochastic matrix** if $P \geq 0$ and each row sum is equal to 1, i.e.,

$$\sum_{j=1}^{n} p_{ij} = 1, \quad i = 1, 2, \ldots, n \;\Leftrightarrow\; Pe = e, \; e = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

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In Markov chains P is called a **transition matrix**.

A **probability distribution vector** is a vector $p \geq 0$ s.t. $e^T p = 1 \Leftrightarrow \sum_{i=1}^{n} p_i = 1$.

Markov Chains

Let $p^{(0)}$ be a given probability distribution vector. We want to know the behaviour of $p^{(k)} = P^T p^{(k-1)} = \cdots = (P^T)^k p^{(0)}$ as $k \to \infty$.

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Theorem (Thm.7)

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Proof: P stochastic $\Rightarrow \rho(P) = 1$ and $Pe = e$. Thm.4 applied to $P^T > 0$ with $x = p$ and $y = e$ gives

$$\lim_{k\to\infty} p^{(k)} = \lim_{k\to\infty} (P^T)^k p^{(0)} = \frac{pe^T}{e^T p} p^{(0)} = \frac{e^T p^{(0)}}{e^T p} p = p$$

since $e^T p^{(0)} = e^T p = 1$.

Example 3

Let $Q = (q_{ij}) > 0$, where $q_{ij} =$ fraction of commodity present in region R_j and ship to region R_i, $i, j = 1, \dots, n$.

Suppose there are x_i units in region R_i today with $\sum_{i=1}^{n} x_i = \alpha$.

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Thm.7 with $p^{(0)} = x/\alpha \Rightarrow \lim_{k\to\infty} \alpha Q^k (x/\alpha) = \alpha p$ with $p \geq 0$ s.t. $Qp = p$, $e^T p = 1$.

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Limit depends only on α and Q and not on x. Verify that $p = [\,6 \quad 16 \quad 11 \quad 11\,]^T/44$.