MATH36001

Lecturer: **Stefan Güttel**
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Aim: Introduction to matrix analysis (analysis of linear transformations) through the development of essential tools like

- Jordan Canonical Form
- Singular Value Decomposition
- Matrix Functions
- Perron–Frobenius theory

Prerequisites: MATH10202 and 10212 (Linear Algebra).

Textbooks: see course website.

Handouts – Exercises – Solutions

- Handouts available, but missing explanations and examples.
- I will show some “real-world applications” in the lectures. Come to the lectures and don’t miss the fun part!
- Each handout contains exercises; difficult solutions to be discussed in feedback session on Monday 10am.
- Website (linked from Blackboard):
  `http://personalpages.manchester.ac.uk/staff/stefan.guettel/ma/`
- Mid-term test: Wednesday, 11th November 2015.

Matrices

An $m \times n$ **matrix** is an array

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in \mathbb{C}^{m \times n}.$$

a_{ij} is the element in position (i, j).
If $m = n$ the matrix is **square**, otherwise it is **rectangular**.

O_{mn}: $m \times n$ **zero matrix**. I_n: $n \times n$ **identity matrix**.

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Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, 3×2 rectangular matrix, $a_{31} = 5$.

Vectors

A **row vector** $x = [\, x_1 \quad x_2 \quad \cdots \quad x_n \,]$ is a $1 \times n$ matrix.

A **column vector** $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ is an $m \times 1$ matrix.

$\mathbb{R}^m \equiv \mathbb{R}^{m\times 1}$ and $\mathbb{C}^m \equiv \mathbb{C}^{m\times 1}$ denote the vector space of real and complex m-vectors, respectively.
The jth column of I_n is called **jth unit vector**:

$$I_n = [\, e_1 \quad e_2 \quad \cdots \quad e_n \,], \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Submatrices

A **submatrix** of A is any matrix obtained by deleting rows and columns.

A **block matrix**

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1q} \\ \vdots & & \vdots \\ A_{p1} & \cdots & A_{pq} \end{bmatrix}$$

is a partitioning of A into submatrices A_{ij} whose dimensions must be consistent.

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Example: $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$,

$$A_{11} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} 3 \end{bmatrix}.$$

Householder's Notation

Generally, we use

capital letters	A, B, C, Δ, Λ	for matrices,
lower case letters	$a_{ij}, b_{ij}, c_{ij}, \delta_{ij}, \lambda_{ij}$	for matrix elements,
lower case letters	x, y, z, c, g, h	for vectors,
lower case Greek letters	$\alpha, \beta, \gamma, \theta, \pi$	for scalars.

Basic Manipulations with Matrices

Transposition: $(\mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{n\times m})$

$$C = A^T \iff c_{ij} = a_{ji}.$$

A^T has rows and cols interchanged, so it is an $n \times m$ matrix.

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Example: $A = \begin{bmatrix} i & 0 \\ 0 & 2-i \\ 0 & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} -i & 0 & 0 \\ 0 & 2+i & 0 \end{bmatrix}.$

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Properties of transposition:

$$\begin{array}{ll} (A^T)^T = A, & (A^*)^* = A, \\ (\alpha A)^T = \alpha A^T, & (\alpha A)^* = \overline{\alpha} A^*, \\ (AB)^T = B^T A^T, & (AB)^* = B^* A^*. \end{array}$$

Basic Manipulations with Matrices

Addition: $(\mathbb{C}^{m\times n} \times \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}^{m\times n})$

$$C = A + B \iff c_{ij} = a_{ij} + b_{ij}.$$

Scalar-matrix multiplication: $(\mathbb{C} \times \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}^{m\times n})$

$$C = \alpha A \iff c_{ij} = \alpha a_{ij}.$$

Properties of matrix addition:

$A + B = B + A$	commutativity
$(A + B) + C = A + (B + C)$	associativity
$\alpha(A + B) = \alpha A + \alpha B$	distributivity of addition
$(\alpha + \beta)A = \alpha A + \beta A$	distributivity of scalar mult.

Basic Manipulations with Matrices

Matrix-matrix multiplication: $(\mathbb{C}^{m\times r} \times \mathbb{C}^{r\times n} \longrightarrow \mathbb{C}^{m\times n})$

$$C = AB \iff c_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}.$$

Properties of matrix multiplication:

$A(BC) = (AB)C$	associativity
$A(B + C) = AB + AC$	distributivity

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Is matrix multiplication commutative, i.e., is $AB = BA$?

Answer: No!

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

Basic Manipulations with Matrices

Block-matrix-matrix multiplication: The formula

$$C = AB \iff C_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}$$

generalizes to block matrices

$$C = \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{m1} & \cdots & C_{mn} \end{bmatrix},$$

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mr} \end{bmatrix}, B = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{r1} & \cdots & B_{rn} \end{bmatrix},$$

provided the blocks are consistent: $A_{ik} \in \mathbb{C}^{m_i \times r_k}$, $B_{kj} \in \mathbb{C}^{r_k \times n_j}$.

Matrix Powers

If $A \neq O_{nn}$, $A^0 \equiv I$, and for any positive integer,

$$A^k = \overbrace{A \cdots A}^{k \text{ times}} = A^{k-1}A = AA^{k-1}.$$

If $p(z) = c_0 + c_1 z + \cdots + c_k z^k$, then given $A \in \mathbb{C}^{n \times n}$,

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A square matrix is

- **involutory** if $A^2 = I$,
- **idempotent** if $A^2 = A$,
- **nilpotent** if $A^k = O$ for some integer $k > 0$.

Inner and Outer Products

Inner product of $x, y \in \mathbb{C}^n$: $x^* y = \sum_{i=1}^{n} \overline{x_i}\, y_i \in \mathbb{C}.$

- $\sqrt{x^* x}$ is the **length** of x.
- $x^* y = 0$ and $x, y \neq 0 \implies x, y$ are **orthogonal**.
- $x^* y = 0$ and $x^* x = y^* y = 1 \implies x, y$ are **orthonormal**.

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Outer product of $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$:

$$xy^* = \begin{bmatrix} x_1\overline{y_1} & \dots & x_1\overline{y_n} \\ \vdots & & \vdots \\ x_m\overline{y_1} & \dots & x_m\overline{y_n} \end{bmatrix} \in \mathbb{C}^{m\times n}.$$

Orthogonal and Unitary matrices

$Q \in \mathbb{R}^{n \times n}$ is **orthogonal** if $QQ^T = I$ and $Q^T Q = I$.

$U \in \mathbb{C}^{n \times n}$ is **unitary** if $UU^* = U^* U = I$.

If $U = [u_1, \ldots, u_n]$ is unitary (or orthogonal) then

$$u_i^* u_j = \delta_{ij} \quad \text{(Kronecker delta)}.$$

The columns of U are mutually orthogonal and of unit length.

Special Matrices

Diagonal matrix: $D = \mathrm{diag}(\alpha_i) = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{bmatrix}$.

$U = \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{bmatrix}$ is **upper triangular**, U^T **lower triangular**.

$A = \begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{p1} & \cdots & \cdots & A_{pp} \end{bmatrix}$ is **block lower triangular**.

Here the A_{ii} are all square but not necessarily of the same size.

Symmetric and Hermitian Matrices

$A \in \mathbb{R}^{n \times n}$ is a **symmetric matrix** if $A^T = A$;

$A \in \mathbb{C}^{n \times n}$ is a **Hermitian matrix** if $A^* = A$.

Let $A \in \mathbb{C}^{n \times n}$ be Hermitian. Then A is

- **positive definite** if $x^* A x > 0$ for all $0 \neq x \in \mathbb{C}^n$,
- **indefinite** if $(x^* A x)(y^* A y) < 0$ for some $x, y \in \mathbb{C}^n$.

Basic Linear Algebra Definitions

A set of vectors $\{v_i\}$ is **linearly dependent** if $\sum_i \alpha_i v_i = 0$ for some α_i not all zero.

Let $A \in \mathbb{C}^{m \times n}$ then

- **rank**(A) is the maximum number of linearly independent rows or columns of A,
- **range**$(A) = \{y \in \mathbb{C}^m : y = Ax \text{ for some } x \in \mathbb{C}^n\}$,
- **null**$(A) = \{x \in \mathbb{C}^n : Ax = 0\}$.

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If $A = [a_1, a_2, \ldots, a_n]$,

$$\mathrm{range}(A) = \mathrm{span}\{a_1, a_2, \ldots, a_n\},$$
$$\mathrm{rank}(A) = \dim(\mathrm{range}(A)).$$

For any $A \in \mathbb{C}^{m \times n}$, $\mathrm{rank}(A) + \dim(\mathrm{null}(A)) = n.$

Determinants

If $A = [\alpha] \in \mathbb{C}^{1\times 1}$ then $\det(A) = \alpha$.
Expansion in cofactors of $\det(A) \in \mathbb{C}^{n\times n}$:

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(\widehat{A}_{ij}) \quad \text{for any } i,$$

where $\widehat{A}_{ij} \in \mathbb{C}^{(n-1)\times(n-1)}$ is a submatrix of A obtained by deleting the ith row and jth column.

Useful properties:

$$\det(AB) = \det(A)\det(B), \quad \det(\alpha A) = \alpha^n \det(A) \quad (\alpha \in \mathbb{C}).$$

$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ block triangular, $\det(A) = \det(A_{11})\det(A_{22})$.

Inverses

If $A, B \in \mathbb{C}^{n\times n}$ satisfy $AB = I$ then B is the **inverse** of A, written $B = A^{-1}$.
If A^{-1} exists A is **nonsingular**; otherwise A is **singular**.

Also, $(AB)^{-1} = B^{-1}A^{-1}$, $\quad (A^{-1})^T = (A^T)^{-1} = A^{-T}$.

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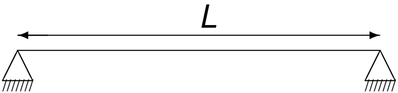
Theorem

For $A \in \mathbb{C}^{n\times n}$ the following conditions are equivalent to A being nonsingular:

1. $\mathrm{null}(A) = \{0\}$ (*i.e., there is no* $0 \neq y \in \mathbb{C}^n$ *s.t.* $Ay = 0$).
2. $\mathrm{rank}(A) = n$ (*i.e., the rows or cols. of A are l.i.*).
3. $\det(A) \neq 0$.
4. *None of A's eigenvalues is zero.*

Beam Problem

Aluminium beam simply supported at both ends:



Transversal displacement $u(x, t)$ governed by a pde

$$\mu \frac{\partial^2 u(x,t)}{\partial t^2} + \kappa \frac{\partial^4 u(x,t)}{\partial x^4} = 0, \quad u(x,t) = u''(x,t) = 0, \; x = 0, L.$$

μ: mass per unit length, κ: bending stiffness.

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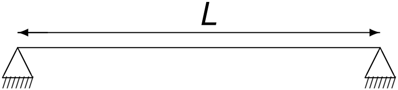

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Separation hypothesis $u(x,t) = e^{i\omega t} v(x)$ yields

$$-\omega^2 \mu\, v(x) + \kappa \tfrac{\partial^4}{\partial x^4} v(x) = 0, \quad v(0) = v''(0) = v(L) = v''(L) = 0.$$

Boundary-value problem for the free vibrations.

Discretized Beam Problem

Finite-difference discretization of $\kappa \frac{\partial^4}{\partial x^4}$ leads to

$$\lambda v = Av. \qquad (*)$$

- $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite (spd) matrix.
- $(*)$ is an eigenvalue problem: λ is an **eigenvalue** and v a corresponding **eigenvector**.

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- $(*)$ is an eigenvalue problem: λ is an **eigenvalue** and v a corresponding **eigenvector**.
- A is spd $\Longrightarrow$ A is **orthogonally diagonalizable:**

$$A = V \Lambda V^T, \text{ with } \Lambda \text{ real} > 0 \text{ diagonal, } V \text{ orth.}$$

- The diagonal elements of Λ are the eigenvalues λ_j and from $\lambda_j = \omega_j^2 \mu$ we calculate the vibration frequencies.
- The columns v_j of V are the corresponding eigenmodes.

First Goals

To study

- **Theory of eigensystems:**
  - eigenvalues, eigenvectors, and invariant subspaces;
  - Schur decomposition, Jordan canonical decomposition;
  - Cayley–Hamilton Theorem;
  - Sylvester's inertia theorem.
- **Norms:**
  - Vector norms and matrix norms,
  - bounds for eigenvalues, Gershgorin theorem.