Aspects of Stone duality for Boolean algebras

A summary of a choice-free version of Stone duality

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Chapter 1

Introduction

In 1936, Stone proved his celebrated representation theorem for Boolean algebras: any Boolean algebra is isomorphic to the clopen sets of its Stone dual, the space of prime filters (see [17]). As Johnstone describes in his introduction to [11], Stone's work has impacted various fields of mathematics, providing us with one of the earliest examples of a non-trivial equivalence of categories, and of a construction of a topological space from purely algebraic data. It has also allowed the study of Boolean algebras from a topological perspective, and research in the same vein as Stone duality has continued since, from Priestley's representation of distributive lattices in [15], to current work by Moshier and Jipsen on bounded lattices using compact open sets which are filters with respect to specialisation in [13] and Bezhanishvili and Holliday on a choice-free version of Stone duality in [3], on which this dissertation is largely based.

Stone's representation requires the use of the Boolean Prime Ideal theorem:

(BPI) Every non-trivial (i.e. with $0 \neq 1$) Boolean algebra has a prime ideal/filter.

which is strictly weaker than AC but not provable within ZF (see [10, Diagram 2.21]).

The reasons we might choose to work without choice principles are detailed by Herrlich in [10]: that a choice-free method for such an important result exists is as good a reason as any to study it, even though it may sacrifice simplicity to avoid choice. In addition, this construction allows us the topological benefits of Stone duality without needing to consider whether choice is actually required.

BPI was necessary in Stone duality to ensure sufficiently many prime filters for the representation to work. To avoid any choice principles then, we should look to avoid these. We consider proper filters instead.

We start the dissertation with some preliminary but important results about the structure of filters in a Boolean algebra. Chapters 3 and 4 detail the construction that Bezhanishvili and Holliday use in [3] for their choice-free duality. Chapter 5 then gives several applications of the duality, also from [3], as well as a non-trivial construction of the dual space of a Boolean algebra. The dissertation then concludes with a discussion of ortholattices, describing their importance in orthologic, and a choice-free duality which parallels that in [3]. This final chapter is largely based on work by Goldblatt in [7, 8] and Yamamoto in [19].

Chapter 2

Preliminaries

Throughout this dissertation we assume basic knowledge of lattices, Boolean algebras and Stone duality, like the type found in [4, Chapters 1, 2, 4, 10, 11] or [9]. We also assume basic category theory that can be found in [12]: categories, functors, natural transformations, and a brief discussion on initial/final objects.

We will often use the notation $\uparrow x = \{y \mid x \leqslant y\}$, where $\leqslant$ is partial order on a Boolean algebra, the specialisation order, or inclusion, depending on context.

Definition 2.1. A *lattice* is a partially ordered set (poset) $(L, \leqslant)$, such that $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist for any $x, y \in L$.

A lattice L is *bounded* if there exist elements $0, 1 \in L$ such that $0 \leqslant x \leqslant 1$ for all $x \in L$.

A lattice L is *distributive* if it satisfies the following for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

An element $x \in L$ is *complemented* if there exists an element $y \in L$ with $x \wedge y = 0$ and $x \vee y = 1$. A lattice is complemented if every element is complemented.

A *Boolean algebra* is a complemented, bounded, distributive lattice. Complements are unique by distributivity, and denoted a'.

Definition 2.2. Let B be a Boolean algebra. A subset F is called a *filter* if

1. whenever $x \in F$ and $x \leqslant y$, then $y \in F$;

2. whenever $x, y \in F$, then $x \wedge y \in F$.

A filter is *prime* if whenever $x \vee y \in F$, either $x \in F$ or $y \in F$. A filter is *maximal* or an *ultrafilter* if it is not strictly contained in any proper filter. In a Boolean algebra these two notions coincide. Note that a proper filter cannot contain both an element and its complement, since otherwise it contains 0 and hence is all of B.

Lower-case letters denote elements of B, and upper-case letters denote subsets of B, filters most of the time. We will refer to the Boolean algebra operations $\vee$ and $\wedge$ as join and meet, respectively.

The following lemmas are important and will be used often in chapters 3, 4 and 6.

Lemma A. *Let B be a Boolean algebra, with $a, b \in B$ such that $0 = a \wedge b$. Then $a \leqslant b'$ and $b \leqslant a'$.*

Proof.

$$\begin{aligned} b' &= b' \vee 0 \\ &= b' \vee (a \wedge b) \\ &= (b' \vee a) \wedge (b' \vee b) \text{ by distributivity} \\ &= (b' \vee a) \wedge 1 \\ &= b' \vee a \end{aligned}$$

So $a \leqslant b'$ and by a similar argument, $b \leqslant a'$. ∎

Lemma B. *Let L be a lattice, $S \subseteq L$ be non-empty. The filter generated by S is the intersection of all filters containing S (arbitrary intersections of filters are filters), and is equal to*

$$\langle S \rangle = \left\{ b \in L \;\middle|\; b \geqslant \bigwedge_{i=1}^{n} a_i \text{ for some } a_i \in S \right\}$$

Proof. Clearly $\langle S \rangle$ is closed upwards. Let $b, \bar{b} \in \langle S \rangle$, so that $b \geqslant \bigwedge_{i=1}^{n} a_i$ and $\bar{b} \geqslant \bigwedge_{j=1}^{m} \bar{a}_j$ for some $a_i, \bar{a}_j \in S$. Then

$$b \wedge \bar{b} \geqslant \bigwedge_{i=1}^{n} a_i \wedge \bigwedge_{j=1}^{m} \bar{a}_j$$

So $b \wedge \bar{b} \in \langle S \rangle$. Any filter containing S must also contain $\langle S \rangle$, so $\langle S \rangle$ is the filter generated by S. ∎

Lemma C. *Let B be a Boolean algebra, and F a proper filter, with $a \notin F$. Now let G be the filter generated by $F \cup \{a'\}$. Then G is proper.*

Proof. We will use the characterisation of G as in Lemma B. Suppose G is not proper. Then $0 \in G$ and $0 \geqslant \bigwedge_{i=1}^{n} b_i$ for some $b_i \in F \cup \{a'\}$. Since 0 is the least element, this is an equality.

Let $J \subseteq \{1, 2, \ldots, n\}$ such that $b_i \in F$ for $i \in J$, and $b_i \in \{a'\}$ for $i \notin J$. Since F is a filter, it is also closed under finite meets, so that $\bigwedge_{i \in J} b_i = c$ for some $c \in F$. Also $\bigwedge_{i \notin J} b_i = a'$. Note that J is both non-empty and a proper subset of $\{1, 2, \ldots, n\}$, since otherwise, $0 = a'$ or $0 \in F$. So $0 = c \wedge a'$. Then $c \leqslant a$ by Lemma A, and as F is a filter with $c \in F$, $a \in F$, a contradiction. So G is proper. ∎

Distributivity is essential for this lemma: see the example $\mathbf{M}_4$ in Chapter 6 for an instance of a complemented, bounded lattice that does not have this property.

Chapter 3

The representation theorems

This chapter develops the first steps of what is necessary to prove Bezhanishvili and Holliday's choice-free version of Stone duality, leaving discussions of the maps to the next chapter. This amounts to finding some subcategory $\mathcal{C}$ of Top, along with functors $\mathsf{BA} \to \mathcal{C}$ and $\mathcal{C} \to \mathsf{BA}$, and then showing that these functors give us the duality, where Top is the category of topological spaces and continuous maps, and BA is the category of Boolean algebras and homomorphisms. So we have four main goals:

(A) Construct a functor $\alpha\colon \mathsf{BA} \to \mathsf{Top}$.

(B) Attempt to categorise the image, $\mathcal{C}$, of this functor in topological terms.

(C) Construct a functor $\beta\colon \mathcal{C} \to \mathsf{BA}$.

(D) Show these two operations are inverse to each other, in the sense that there are isomorphisms $B \to \beta\alpha(B)$ and $X \to \alpha\beta(X)$ for $B \in \mathsf{BA}$ and $X \in \mathcal{C}$.

A key feature of classical Stone duality is that any Boolean algebra is isomorphic to a field of sets, that is, a subalgebra of the powerset algebra, but unfortunately, this property implies BPI. If any Boolean algebra is a field of sets $\mathcal{F} \subseteq \mathcal{P}(X)$, for a set X, then for any $x \in X$, $\{A \in \mathcal{F} \mid x \in A\}$ is a prime filter. So we should expect our Boolean algebra operations to differ from the usual.

The structure of this section is reminiscent of classical Stone duality, with key ideas diverging only when necessary to avoid using BPI.

3.1 The dual space

As mentioned above, our first goal is to construct a topological space from the Boolean algebra B, the dual space.

First we define some topological notions that we will use throughout.

Definition 3.1. Let X be a topological space. Define

- $\mathsf{O}(X)$ as the collection of open subsets of X;

- $\mathsf{CO}(X)$ as the collection of compact open subsets of X;
- For $x \in X$, $\star(x)$ as the set of elements of $\star(X)$ which contain x, where $\star$ is one of $\mathsf{O}, \mathsf{CO}, \mathcal{RO}, \mathsf{CO}\mathcal{RO}$ - these last two will be defined later.

We are now able to define what will be the dual space of a Boolean algebra B.

Definition 3.2. Let B be a Boolean algebra. Define $UV(B)$ to be the space of proper filters on B with topology generated by $\{\hat{a} \mid a \in B\}$, where $\hat{a} = \{F \in \mathrm{PropFilt}(B) \mid a \in F\}$. Note that $\hat{a} \cap \hat{b} = \widehat{a \wedge b}$, so $\{\hat{a} \mid a \in B\}$ is closed under intersection. We will call this the spectral topology on $UV(B)$, to distinguish from another topology defined soon.

This definition is similar to the dual space in classical Stone duality, the only exception being the use of proper filters instead of prime filters; a necessary change given our avoidance of BPI.

3.2 Spectral spaces

We now wish to categorise completely the type of space that $UV(B)$ is, as in goal (B). In [18], Stone used spectral spaces as the dual space of bounded distributive lattices. [5, Theorem 1.3.4] also shows that Stone spaces are precisely Hausdorff spectral spaces. We might then suspect that spectral spaces play an important role in a choice-free duality. We investigate whether this is the case in this section.

Definition 3.3. A topological space X is a *spectral space* if:

1. X is compact;
2. T_0 (for any distinct elements of X, there is an open set containing one but not the other);
3. coherent ($\mathsf{CO}(X)$ is closed under intersection and forms a basis for the topology);
4. sober (every completely prime filter in $\mathsf{O}(X)$ ordered by inclusion is $\mathsf{O}(x)$ for some $x \in X$).

Example. Every finite T_0 space is spectral. See [5, Proposition 1.1.15]. The apparent difference in this definition and the one in [5] is explained in 1.1.14.

Definition 3.4. Let X be a T_0 space. Define the *specialisation order* on X by:

$$x \leqslant y \text{ if and only if } y \text{ is contained in every open set that contains } x.$$

Note that if $x \leqslant y$ and $y \leqslant x$, then x and y are contained in precisely the same open sets, so as X is T_0, $x = y$. Clearly $\leqslant$ is transitive, and so is indeed a weak partial order on X.

The following proposition proves that this dual space is a spectral space and establishes a useful result about the specialisation order of the dual space.

Proposition 3.5. *For a Boolean algebra B:*

(i) *$UV(B)$ is a spectral space;*

(ii) *the specialisation order in $UV(B)$ is the inclusion order.*

Proof. We first show that each $\hat{a}$ is compact in $UV(B)$. So suppose $\mathcal{U}$ is an open cover for $\hat{a}$. Since $\{\hat{b} \mid b \in B\}$ is a basis for the topology, we can assume $\mathcal{U} = \{\hat{b}_i \mid i \in I\}$ and so $\hat{a} \subseteq \bigcup_{i\in I} \hat{b}_i$. Then any proper filter containing a must also contain one of the elements b_i. Consider the filter $\uparrow a$. This filter is proper (otherwise, a is 0 and so $\hat{a}$ is empty and trivially compact), and contains a. Then $\uparrow a$ contains some b_i, and $a \leqslant b_i$. So any proper filter containing a must also contain b_i, and so $\hat{a} \subseteq \hat{b}_i$. That is, $\hat{a}$ is compact.

Any compact open set in $UV(B)$ has the form $\bigcup_{i\in I} \hat{b}_i$ for finite I. Then

$$\bigcup_{i\in I} \hat{a}_i \;\cap\; \bigcup_{j\in J} \hat{b}_j = \bigcup_{i\in I, j\in J} (\hat{a}_i \cap \hat{b}_j) = \bigcup_{i\in I, j\in J} \widehat{a_i \wedge b_j}$$

This is a finite union of compact open sets so is compact and open. Clearly this generates the same topology as $\{\hat{a} \mid a \in B\}$ and so is also a basis. Then $UV(B)$ is coherent.

Suppose F and G are distinct, proper filters on B. Without loss of generality, assume $F \nsubseteq G$ and let $a \in F \setminus G$. Then $\hat{a}$ is an open set containing F but not G, and $UV(B)$ is T_0.

Now let $\mathcal{F}$ be a completely prime filter in $\mathsf{O}(UV(B))$. Let F be the filter generated by $\{a \in B \mid \hat{a} \in \mathcal{F}\}$. $\mathcal{F}$ is proper so F is proper. We show that $\mathcal{F} = \{U \in \mathsf{O}(UV(B)) \mid F \in U\}$. Let U be any open set in $\mathcal{F}$. Then $U = \bigcup_{i\in I} \hat{a}_i$, and as $\mathcal{F}$ is a completely prime filter, $\hat{a}_j \in \mathcal{F}$ for some j. Then $a_j \in F$, so $F \in \hat{a}_j \subseteq U$. For the other direction, suppose U is an open set containing F. Then $U = \bigcup_{i\in I} \hat{a}_i$, so $F \in \hat{a}_j$ for some $j \in I$. Then $a_j \in F$. By Lemma B, this means that $a_j \geqslant \bigwedge_{i=1}^{n} b_i$ for $\hat{b}_i \in \mathcal{F}$. As $\mathcal{F}$ is a filter, $\bigcap_{i=1}^{n} \hat{b}_i \in \mathcal{F}$, so $\hat{a}_j \in \mathcal{F}$ and since $U \supseteq \hat{a}_j$, $U \in \mathcal{F}$.

Then $UV(B)$ is sober, and so a spectral space.

In proving $UV(B)$ is T_0, we saw that if $F \nsubseteq G$, then there is an open set containing F but not G, so that, $F \nleqslant G$, where $\leqslant$ is the specialisation order on $UV(B)$. Now suppose $F \subseteq G$. Let $U = \bigcup_{i\in I} \hat{a}_i$ be any open set containing F. Then $F \in \hat{a}_j$ for some $j \in I$, so $a_j \in F$. As $F \subseteq G$, $a_j \in G$ and $G \in \hat{a}_j \subseteq U$. Then $F \leqslant G$. ■

3.3 The inverse operation

Having settled on this $UV(B)$ being the dual space, we now need an operation taking $UV(B)$ back isomorphically to B. In classical Stone duality, this is done by taking the clopen sets of the dual Stone space.

Definition 3.6. Let X be a space. U is a *regular open* set if $U = \mathsf{int}(\mathsf{cl}(U))$.

Example. In $\mathbb{R}^2$, regular open sets can be thought of as those without any 'cracks' ([9, page 14]. $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ is regular open but $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ is not.

Definition 3.7. Let X be a space with specialisation order $\leqslant$. Define $\mathsf{Up}(X,\leqslant)$ to be the topology generated by the upsets of $(X,\leqslant)$, that is, the sets that are closed upwards under $\leqslant$. Then define

- $\mathcal{RO}(X)$ to be the set of regular open sets in $(X, \mathsf{Up}(X,\leqslant))$;
- $\mathsf{CO}\mathcal{RO}(X) = \mathsf{CO}(X) \cap \mathcal{RO}(X)$.

Note that upsets are closed under arbitrary unions and intersections, so that $\mathsf{Up}(X,\leqslant)$ consists precisely of upsets of X.

The following is a technical lemma that will be used throughout this section.

Lemma 3.8. $\mathsf{int}_{\leqslant}(\mathsf{cl}_{\leqslant}(U)) = \{x \in X \mid \forall y \geqslant x, \exists z \geqslant y \textit{ such that } z \in U\}$.

Proof. We will drop the $\leqslant$ subscript from int and cl for ease of notation, but note that at no point do we make use of the original topology.

$\mathsf{int}(\mathsf{cl}(U)) = \mathsf{int}(X \setminus \mathsf{int}(X \setminus U))$. Let $x \in \mathsf{int}(\mathsf{cl}(U))$. Then there is some upset $A \subseteq X \setminus \mathsf{int}(X \setminus U)$ with $x \in A$. Let $y \geqslant x$. We need to show there is $z \geqslant y$ with $z \in U$. So suppose not. Then $\uparrow y \subseteq X \setminus U$. Also, A is an upset, so $y \in A$, and so $y \notin \mathsf{int}(X \setminus U)$. But then $\uparrow y$ is an upset contained in $X \setminus U$, and so it is contained in $\mathsf{int}(X \setminus U)$, a contradiction since $y \in \uparrow y$.

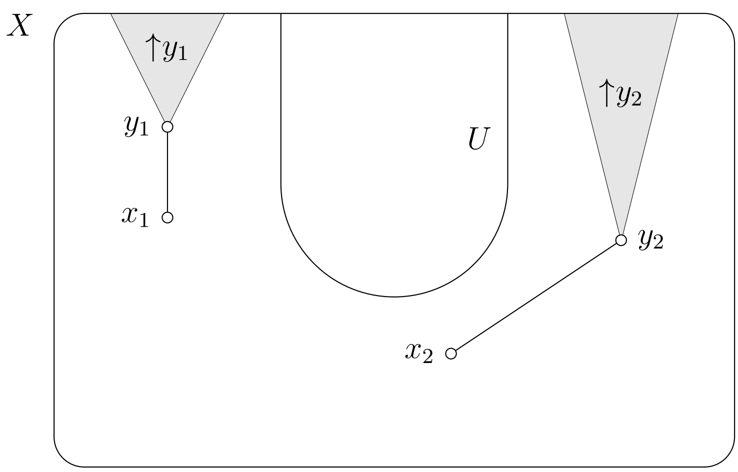
Now let x be an element of the right hand side. We show that $\uparrow x \subseteq X \setminus \mathsf{int}(X \setminus U)$. So let $y \geqslant x$.

Claim. $y \notin \mathsf{int}(X \setminus U)$

Proof of claim. Let A be any upset contained in $X \setminus U$. Suppose $y \in A$. Then as A is an upset, for any $z \geqslant y$, $z \in A$, so $z \notin U$, but this contradicts x being an element of the right hand side. So y is not in any upset contained in $X \setminus U$, and hence $y \notin \mathsf{int}(X \setminus U)$.

Then $\uparrow x$ is an upset contained in $X \setminus \mathsf{int}(X \setminus U)$, and hence is contained in $\mathsf{int}(X \setminus \mathsf{int}(X \setminus U))$, and so x is an element of the left hand side. ■

Remark. U is regular open in the upset topology if and only if $U = \{x \in X \mid \forall y \geqslant x, \exists z \geqslant y$ such that $z \in U\}$. The following diagram gives a intuitive idea for what a set looks like if it is regular open in the upset topology.



3.4 The representation theorem for Boolean algebras

Now we are able to present the choice-free representation theorem, though more work must be done to show this is a duality.

Theorem 3.9. *Let B be a Boolean algebra with order $\leqslant$. Then there is an isomorphism $\phi\colon B \to \mathsf{CO}\mathcal{RO}(UV(B))$ where $\mathsf{CO}\mathcal{RO}(UV(B))$ is ordered by inclusion. $\mathsf{CO}\mathcal{RO}(UV(B))$ is a Boolean algebra with operations given by:*

- $U \wedge V = U \cap V$
- $U \vee V = \mathsf{int}_{\leqslant}(\mathsf{cl}_{\leqslant}(U \cup V))$
- $\neg U = \mathsf{int}_{\leqslant}(UV(B) \setminus U)$

Proof. Let $\phi(a) = \hat{a}$. We need to show $\mathsf{CO}\mathcal{RO}(UV(B)) = \{\hat{a} \mid a \in B\}$, but first we establish a useful result that will help to prove this.

Claim 1. $\widehat{\bigvee_{i=1}^n a_i} = \mathsf{int}_{\leqslant}(\mathsf{cl}_{\leqslant}(\bigcup_{i=1}^n \hat{a}_i))$.

Proof of claim. Let $F \in \widehat{\bigvee_{i=1}^n a_i}$. Then $\bigvee_{i=1}^n a_i \in F$. Let $G \supseteq F$ (equivalent to $G \geqslant F$). We want to show there is $H \supseteq G$ such that $H \in \bigcup_{i=1}^n \hat{a}_i$. If for each i, $a_i' \in G$, then $\bigwedge_{i=1}^n a_i' = (\bigvee_{i=1}^n a_i)' \in G$, so G is not proper. So say $a_j' \notin G$. Let H be the filter generated by $G \cup \{a_j\}$. H is proper by Lemma C, contains G, and $H \in \bigcup_{i=1}^n \hat{a}_i$. By the characterisation mentioned earlier, this gives the left to right inclusion. Now suppose $F \notin \widehat{\bigvee_{i=1}^n a_i}$, so that $\bigvee_{i=1}^n a_i \notin F$. Then the filter G generated by $F \cup \{\bigwedge_{i=1}^n a_i'\}$ is proper by Lemma C and if any $a_j \in G$, then $\bigvee_{i=1}^n a_i \in G$. But also $(\bigvee_{i=1}^n a_i)' \in G$, so G is not proper. Then any proper filter $H \supseteq G$ contains no a_i, so that $H \notin \bigcup_{i=1}^n \hat{a}_i$.

Now we can show that $\mathsf{CO}\mathcal{RO}(UV(B)) = \{\hat{a} \mid a \in B\}$. We showed before that each $\hat{a}$ is compact and open in $UV(B)$. Using the characterisation of regular open sets given earlier, we need to show $\hat{a} = \{F \in \text{PropFilt}(B) \mid \forall G \geqslant F, \exists H \geqslant G \text{ such that } H \in \hat{a}\}$, where $\leqslant$ is specialisation, or equivalently, inclusion. Then clearly $\hat{a}$ is an upset, giving the left to right inclusion. For the converse, take $F \notin \hat{a}$, so that $a \notin F$. The filter G generated by $F \cup \{a'\}$ is a proper filter by Lemma C, contains F, and for any proper filter H containing G, $a \notin H$, and so $H \notin \hat{a}$. This gives the right to left inclusion.

For $\mathsf{CO}\mathcal{RO}(UV(B)) \subseteq \{\hat{a} \mid a \in B\}$, let $S \in \mathsf{CO}\mathcal{RO}(UV(B))$. As S is compact open, it is a finite union of $\hat{a}$, and so $S = \bigcup_{i=1}^n \hat{a}_i$. Then since S is regular open, and using Claim 1,

$$S = \mathsf{int}_{\leqslant}(\mathsf{cl}_{\leqslant}(S)) = \widehat{\bigvee_{i=1}^n a_i}$$

and so $\mathsf{CO}\mathcal{RO}(UV(B)) = \{\hat{a} \mid a \in B\}$.

It is easy to see that ϕ preserves the order, and so is then an order isomorphism, with $\hat{a} \wedge \hat{b} = \widehat{a \wedge b}$, $\hat{a} \vee \hat{b} = \widehat{a \vee b}$, and $\neg\hat{a} = \widehat{a'}$. We have already shown that $\widehat{a \wedge b} = \hat{a} \cap \hat{b}$, and Claim 1 gives $\vee$ on $\mathsf{CO}\mathcal{RO}(UV(B))$.

Then it remains to show that $\widehat{a'} = \mathsf{int}_{\leqslant}(UV(B) \setminus \hat{a})$. Let $F \in \widehat{a'}$, so that $a' \in F$. Then as F is a proper filter, $a \notin F$ so $F \in UV(B) \setminus \hat{a}$. Now let $G \supseteq F$ be a proper filter.

Then also $a' \in G$, so $a \notin G$, and $G \in UV(B) \setminus \hat{a}$. Therefore $\uparrow F$ is an upset contained in $UV(B) \setminus \hat{a}$, so $F \in \mathsf{int}_{\leqslant}(UV(B) \setminus \hat{a})$. Now let $F \notin \widehat{a'}$, so that $a' \notin F$. The filter G generated by $F \cup \{a\}$ is proper by Lemma C and $F \subseteq G \in \hat{a}$. Then any upset containing F contains G but $G \notin UV(B) \setminus \hat{a}$. So $F \notin \mathsf{int}_{\leqslant}(UV(B) \setminus \hat{a})$. ■

3.5 UV spaces

In the previous section we have managed to represent every Boolean algebra as a set of subsets of a particular topological space. To go further, we would like to show that every such space can be obtained from a Boolean algebra in this way. However, as the following example shows, this is not possible with spectral spaces.

Example. Let $X = \{a, b\}$ be a space with T_0 topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$. We have already mentioned that finite T_0 spaces are spectral.

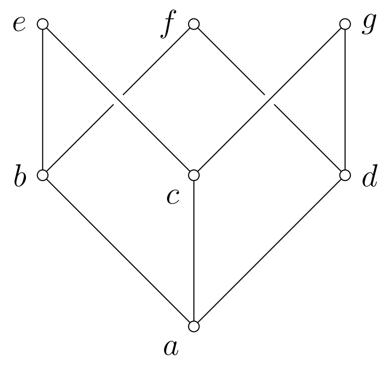
Suppose there was a Boolean algebra B such that $UV(B) \cong X$. Then B has two proper filters. For every non-zero element, c, of a Boolean algebra, the set $\uparrow c$ is a proper filter. So B has at most two non-zero elements. Finite Boolean algebras have 2^n elements for some $n \geqslant 0$ (see [4, Section 5.6]). Then B must have one or two elements, but in either case, there are not two proper filters. So X is not $UV(B)$ for any Boolean algebra B.

So we must restrict to a proper subclass of spectral spaces. The following definition introduces UV spaces, which turn out to be just what is necessary.

Definition 3.10. Let X be a topological space. Then X is called a *UV space* if:

1. X is T_0;
2. $\mathsf{CO}\mathcal{RO}(X)$ is closed under $\cap$ and $\mathsf{int}_{\leqslant}(X \setminus \cdot)$, and forms a basis for the topology[1];
3. every proper filter in $\mathsf{CO}\mathcal{RO}(X)$ is $\mathsf{CO}\mathcal{RO}(x)$ for some $x \in X$.

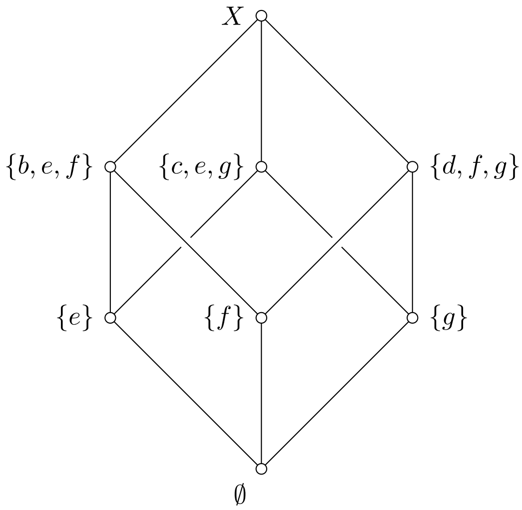
Example. Here we give a simple example of a UV space, and the Boolean algebra from which it can be constructed. Let X be a space with seven elements, $\{a, b, c, d, e, f, g\}$, and specialisation order given by the following poset diagram:



[1]This is the updated definition in [3, Definition 5.1].

X is T_0 since this is a non-trivial partial order. For finite spaces, the topology is uniquely determined by the specialisation order. To see this note that a subset is open if and only if it is an upset.

X is finite so every subset is compact. Using the remark after Lemma 3.8, we find that the following sets are precisely the elements of $\mathsf{CO}\mathcal{RO}(X)$.



From this diagram it is easy to see that X satisfies conditions 2 and 3, and so is a UV space. Note that $\mathsf{CO}\mathcal{RO}(X)$ is a Boolean algebra, and can be obtained from X by adding a top element. This fact is characteristic of all finite UV spaces, and is proved in [3, Corollary 5.4.5].

Theorem 3.11. *For a Boolean algebra B, $UV(B)$ is a UV space.*

Proof. In Theorem 3.9 and Proposition 3.5, we showed that $UV(B)$ is spectral and that $\mathsf{CO}\mathcal{RO}(UV(B))$ is a Boolean algebra. $UV(B)$ has $\{\hat{a} \mid a \in B\} = \mathsf{CO}\mathcal{RO}(UV(B))$ as a basis for its topology. This proves conditions 1 and 2.

Let $\mathcal{F}$ be a proper filter in $\mathsf{CO}\mathcal{RO}(UV(B))$. Let F be $\{a \in B \mid \hat{a} \in \mathcal{F}\}$. This is a proper filter by the isomorphism in Theorem 3.9. We claim $\mathsf{CO}\mathcal{RO}(F) = \mathcal{F}$. $\mathsf{CO}\mathcal{RO}(F) = \{U \in \mathsf{CO}\mathcal{RO}(UV(B)) \mid F \in U\}$. Then $\hat{a} \in \mathcal{F} \Leftrightarrow a \in F \Leftrightarrow F \in \hat{a}$. So $\mathsf{CO}\mathcal{RO}(F) = \mathcal{F}$. ■

Proposition 3.12. *For a UV space X, $\mathsf{CO}\mathcal{RO}(X)$ is a Boolean algebra ordered by inclusion with operations:*

- $U \wedge V = U \cap V$
- $U \vee V = \mathsf{int}_{\leqslant}(\mathsf{cl}_{\leqslant}(U \cup V))$
- $\neg U = \mathsf{int}_{\leqslant}(X \setminus U)$

Proof. See [9, Section 4, Theorem 1] for a proof that the regular open sets of any space form a Boolean algebra with the operations defined above. Then condition 2 of Definition 3.10 ensures $\mathsf{CO}\mathcal{RO}(X)$ is a subalgebra of this.

To see the order on this Boolean algebra is inclusion, note that the order is given by $a \leqslant b \Leftrightarrow a = a \wedge b$. Once we establish $U \wedge V = U \cap V$, then $U \leqslant V \Leftrightarrow U \cap V = U \Leftrightarrow U \subseteq V$. ■

3.6 The representation theorem for UV spaces

Theorem 3.13. *Let X be a UV space. Then $\theta\colon X \to UV(\mathsf{CO}\mathcal{RO}(X))$ where $x \mapsto \mathsf{CO}\mathcal{RO}(x) = \{U \in \mathsf{CO}\mathcal{RO}(X) \mid x \in U\}$ is a homeomorphism.*

Proof. First, since $\mathsf{CO}\mathcal{RO}(X)$ is closed under intersection, and $\emptyset \in \mathsf{CO}\mathcal{RO}(X)$, $\mathsf{CO}\mathcal{RO}(x)$ is a proper filter.

Injective. Suppose $x \neq y$ for $x, y \in X$. Since X is T_0, there is an open set containing one but not the other, and since $\mathsf{CO}\mathcal{RO}(X)$ forms a basis, there is some $U \in \mathsf{CO}\mathcal{RO}(X)$ containing one but not the other. So $\mathsf{CO}\mathcal{RO}(x) \neq \mathsf{CO}\mathcal{RO}(y)$.

Surjective. This is guaranteed by condition 3 of Definition 3.10.

Continuous. Is it enough to check θ is continuous on elements of the basis of the topology. Such an open set is $\hat{U}$ for $U \in \mathsf{CO}\mathcal{RO}(X)$. Then

$$\begin{aligned}\theta^{-1}[\hat{U}] &= \{x \in X \mid \mathsf{CO}\mathcal{RO}(x) \in \hat{U}\}\\ &= \{x \in X \mid U \in \mathsf{CO}\mathcal{RO}(x)\}\\ &= \{x \in X \mid x \in U\}\\ &= U\end{aligned}$$

Continuous inverse. Similarly it is enough to check θ^{-1} is continuous on elements of the basis of the topology. So let $U \in \mathsf{CO}\mathcal{RO}(X)$ be a basic open set in X. We need to show that $\theta[U]$ is open.

$$\begin{aligned}\theta[U] &= \{\mathsf{CO}\mathcal{RO}(x) \mid x \in U\}\\ &= \{\mathsf{CO}\mathcal{RO}(x) \mid U \in \mathsf{CO}\mathcal{RO}(x)\}\\ &= \{\mathsf{CO}\mathcal{RO}(x) \mid \mathsf{CO}\mathcal{RO}(x) \in \hat{U}\}\\ &= \hat{U}\end{aligned}$$

This last equality follows from the fact that the sets $\mathsf{CO}\mathcal{RO}(x)$ are precisely the proper filters. ∎

As a corollary, we get:

Corollary 3.14. *Let X be a UV space. Then X is a spectral space.*

Proof. X is homeomorphic to $UV(\mathsf{CO}\mathcal{RO}(X))$, $\mathsf{CO}\mathcal{RO}(X)$ is a Boolean algebra, and $UV(B)$ is spectral if B is a Boolean algebra. ∎

3.7 Regular open sets in the two topologies

In the case of compact open sets of $UV(B)$, being regular open in the upset topology is the same as being regular open in the spectral topology. Let $\mathsf{RO}(UV(B))$ be the sets which are regular open in the spectral topology, and $\mathsf{CRO}(UV(B))$ the compact sets in $\mathsf{RO}(UV(B))$. Recall that $\neg U = \mathsf{int}_{\leqslant}(UV(B) \setminus U)$ and define $U^* = \mathsf{int}(UV(B) \setminus U)$. Then $U \in \mathsf{RO}(UV(B))$ if and only if $U = U^{**}$. Then

$$U^* = \bigcup\{V \in \mathsf{O}(UV(B)) \mid U \cap V = \emptyset\} = \bigcup\{\hat{a} \mid U \cap \hat{a} = \emptyset\}$$

Proposition 3.15. *Let B be a Boolean algebra. Then:*

(i) *if $U \in \mathsf{O}(UV(B))$, then $U^* \subseteq \neg U$;*

(ii) *if $U \in \mathsf{CO}(UV(B))$, then $\neg U \subseteq U^*$.*

Proof. For (i), if $F \in U^*$, then $F \in \hat{a}$ and $U \cap \hat{a} = \emptyset$ for some $a \in B$. Then any proper filter containing F also contains a so cannot be in U. So $F \in \neg U$.

For (ii), let $U = \bigcup_{i=1}^{n} \hat{a}_i$ and $F \in \neg U$. If some $a'_j \notin F$ then the filter generated by $F \cup \{a_j\}$ is proper by Lemma C, contains F and is an element of U, contradicting $F \in \neg U$. So $c = \bigwedge_{i=1}^{n} a'_i \in F$. Then $F \in \hat{c}$ and $U \cap \hat{c} = \emptyset$, so $F \in U^*$. ■

From this we have

Corollary 3.16. $\mathsf{CO}\mathcal{RO}(UV(B)) = \mathsf{CRO}(UV(B))$*, and so* $B \cong \mathsf{CRO}(UV(B))$.

If we are not in the case of compact opens, $\neg$ and * may differ, as the following ([3, Proposition 4.3]) demonstrates.

Proposition 3.17. *Let B be a Boolean algebra.*

(i) $\mathsf{RO}(UV(B) \subseteq \mathsf{O}\mathcal{RO}(UV(B))$*;*

(ii) *Let F be a non-principal ultrafilter in B and $U = \bigcup\{\widehat{a'} \mid a \in F\}$. Then*

  (a) *$F \in \neg U \setminus U^*$;*

  (b) *$U = \neg\neg U$;*

  (c) *$U \subsetneq U^{**}$.*

(iii) *Let F is a principal filter in B and $U \in \mathsf{O}(UV(B))$. If $F \in \neg U$ then $F \in U^*$.*

Assuming BPI, every infinite Boolean algebra has a non-principal ultrafilter (see [6, page 174]). Then, from part (ii), $\neg$ and * can be distinguished with an open set. Also, it is consistent with ZF that there exists an infinite Boolean algebra with no non-principal filters (see [14, Proposition 2.5]), so by part (iii) and Proposition 3.15, $\neg$ and * cannot be distinguished by open sets.

Chapter 4

Choice-free duality

So far, we have established operations between Boolean algebras and UV spaces, that is, we have defined our functors on objects. To extend this correspondence to an equivalence, we need to consider the structure-preserving maps between them. For Boolean algebras, this is the Boolean algebra homomorphism: a map of the underlying sets $A \to B$ that preserves the operations. To such a map, we can associate a canonical map in the other direction between their corresponding UV spaces; the preimage of a proper filter in B is a proper filter in A. The preimage map is also used in classical Stone duality. We then have the following new goals:

(E) Categorise in topological terms these maps $UV(B) \to UV(A)$ arising from Boolean algebra homomorphisms $A \to B$.

(F) Associate to any such map of UV spaces $X \to Y$ a corresponding Boolean algebra homomorphism $\mathsf{CO}\mathcal{RO}(Y) \to \mathsf{CO}\mathcal{RO}(X)$.

(G) Show these operations on maps are inverse to each other, in the sense of a categorical duality.

As the following proposition shows, we should not expect the maps $UV(B) \to UV(A)$ to be just continuous maps, like in the case of classical Stone duality.

Proposition 4.1. *If $\mathcal{C}$ is some full subcategory of* Top *(that is, the morphisms are precisely those in* Top*), and is dually equivalent to* BA*, then BPI holds.*

Proof. BA has a unique initial object $\mathbf{2} = \{0, 1\}$. So $\mathcal{C}$ has a unique final object X, and the duality maps these objects to each other. As $\mathcal{C}$ is a full subcategory of Top, and Top contains a final object $\mathbb{1}$, the space with one element, $X \cong \mathbb{1}$. For any non-empty $Y \in \mathcal{C}$, there is a continuous map $\mathbb{1} \to Y$, and so there must be a homomorphism from any non-trivial Boolean algebra to $\mathbf{2}$. Any such homomorphism h gives a prime filter $h^{-1}[1]$, and so any non-trivial Boolean algebra has a prime filter. Note that the empty space is initial in Top, and the 1-element Boolean algebra $\mathbf{1}$ is final in BA, so the duality maps these to each other. Then any non-trivial Boolean algebra comes from a non-empty space in Top. ■

4.1 Morphisms

Definition 4.2. Let A, B be Boolean algebras. A map $f : A \to B$ is a *(Boolean algebra) homomorphism* if it preserves $\wedge, \vee$, and $'$. That is, for $a, b \in A$:

- $f(a \wedge b) = f(a) \wedge f(b)$;
- $f(a \vee b) = f(a) \vee f(b)$;
- $f(a') = f(a)'$.

Note that these conditions imply f must preserve 0 and 1.

Proposition 4.3. *Let A, B be Boolean algebras with $h : A \to B$ a homomorphism. Then for a proper filter $F \subseteq B$, the map $h_+ : UV(B) \to UV(A)$ sending $F \mapsto h^{-1}[F]$ satisfies the following condition:*

$$h_+^{-1}[U] \in \mathsf{CO}(UV(B)) \textit{ whenever } U \in \mathsf{CO}(UV(A)).$$

Proof. First, if F is a proper filter in B, then $h_+(F) = h^{-1}[F]$ is a proper filter in A. Let U be compact open in $UV(A)$. Then $U = \bigcup_{i=1}^{n} \hat{a}_i$. Then $h_+^{-1}[U] = \bigcup_{i=1}^{n} h_+^{-1}[\hat{a}_i]$, and

$$\begin{aligned}
h_+^{-1}[\hat{a}_i] &= \{G \in UV(B) \mid h_+(G) \in \hat{a}_i\} \\
&= \{G \in UV(B) \mid h^{-1}[G] \in \hat{a}_i\} \\
&= \{G \in UV(B) \mid a_i \in h^{-1}[G]\} \\
&= \{G \in UV(B) \mid h(a_i) \in G\} \\
&= \widehat{h(a_i)}
\end{aligned}$$

Then $h_+^{-1}[U] = \bigcup_{i=1}^{n} \widehat{h(a_i)}$ is a finite union of compact open sets, so is compact open. ■

Any map of spectral spaces satisfying this condition is said to be a *spectral map*. Let Spec be the category of spectral spaces with spectral maps. In Proposition 4.1, it was proved that if BA was dually equivalent to a full subcategory of Top then BPI holds. All the spaces and maps mentioned in that proof are actually spectral, so the same argument holds for a full subcategory of Spec. That these spaces and functions are indeed spectral is discussed in [5, Sections 1.2.5-7].

Therefore, in order to develop the duality for the functors we have already chosen, we must look for an additional property for the morphisms of the dual category. The following proposition shows that h_+ satisfies the *p-morphism* condition, and this turns out to be exactly what is required.

Proposition 4.4. *Let A, B be Boolean algebras with $h : A \to B$ a homomorphism. Then the map h_+ satisfies the following condition:*

$$\textit{if } h_+(F) \subseteq \tilde{F} \textit{ then there is } G \in UV(B) \textit{ such that } F \subseteq G \textit{ and } h_+(G) = \tilde{F}.$$

Proof. Suppose $h_+(F) \subseteq \tilde{F}$. Let G be the filter generated by $F \cup h[\tilde{F}]$. Suppose for a contradiction that G is not proper. Then by Lemma B, and since 0 is the least element, there are $b_i \in F \cup h[\tilde{F}]$ such that $0 = \bigwedge_{i=1}^{n} b_i$. Since $F, \tilde{F}$ are filters so are closed under finite meets, and h is a homomorphism, we can combine the corresponding b_i to get $0 = a \wedge h(c)$, for $a \in F$ and $c \in \tilde{F}$. Then by Lemma A, $a \leqslant h(c)' = h(c')$. As $a \in F$, $h(c') \in F$, and so $c' \in h_+(F) \subseteq \tilde{F}$. Then $c \wedge c' = 0 \in \tilde{F}$, a contradiction. So G is proper and $F \subseteq G$. It remains to check $h_+(G) = \tilde{F}$.

By construction we have $h[\tilde{F}] \subseteq G$, so $\tilde{F} \subseteq h_+(G)$. Let $a \in h_+(G)$, so $h(a) \in G$. Then, again by Lemma B, $h(a) \geqslant b \wedge h(c)$ for $b \in F$ and $c \in \tilde{F}$. Then

$$\begin{aligned} h(a) \vee h(c)' &\geqslant (b \wedge h(c)) \vee h(c)' \\ &= (b \vee h(c)') \wedge (h(c) \vee h(c)') \text{ by distributivity} \\ &= (b \vee h(c)') \wedge 1 \\ &= b \vee h(c)' \\ &\geqslant b \end{aligned}$$

Then $b \leqslant h(a) \vee h(c)' = h(a \vee c')$. As $b \in F$, $h(a \vee c') \in F$, so $a \vee c' \in h_+(F) \subseteq \tilde{F}$. Also $c \in \tilde{F}$, so $(a \vee c') \wedge c \in \tilde{F}$, and

$$(a \vee c') \wedge c = (a \wedge c) \vee (c' \wedge c) = a \wedge c \leqslant a$$

so $a \in \tilde{F}$. Then $h_+(G) = \tilde{F}$, so h_+ satisfies the condition. ■

Definition 4.5. Let X, Y be UV spaces. A map $f\colon X \to Y$ is a *UV map* if it is a spectral map and satisfies the p-morphism condition:

1. $f^{-1}[U] \in \mathsf{CO}(X)$ whenever $U \in \mathsf{CO}(Y)$;
2. if $f(x) \leqslant \tilde{y}$ then there is $y \in X$ such that $x \leqslant y$ and $f(y) = \tilde{y}$.

So we have shown that h_+ is a UV map.

Remark. Condition 1 implies f is continuous, since a UV space is spectral and spectral spaces have $\mathsf{CO}(X)$ as a basis of the topology.

For the duality to work, for each UV map, there needs to be an associated map of the dual Boolean algebras that preserve the operations. Since spectral maps preserve compact open sets under preimage, we might hope UV maps preserve compact open, regular open sets under preimage.

Proposition 4.6. *Let X, Y be UV spaces with $f\colon X \to Y$ a UV map. Then for $U \in \mathsf{CO}\mathcal{RO}(Y)$ the following hold:*

(i) $f^{-1}[U] \in \mathsf{CO}\mathcal{RO}(X)$;

(ii) $f^{-1}[\mathsf{int}_{\leqslant}(Y \setminus U)] = \mathsf{int}_{\leqslant}(X \setminus f^{-1}[U])$.

Proof. First we show that $f^{-1}[U]$ is open in the upset topology on X. Let $x \in f^{-1}[U]$, and $y \geqslant x$. Then since f preserves the specialisation order, $f(x) \leqslant f(y)$, and $f(x) \in U$. As U is an upset, $f(y) \in U$, and so $y \in f^{-1}[U]$.

To show $f^{-1}[U]$ is regular open in the upset topology, we need to show that $f^{-1}[U] = \{x \in X \mid \forall y \geqslant x, \exists z \geqslant y \text{ such that } z \in f^{-1}[U]\}$. So let $x \notin f^{-1}[U]$. Then $f(x) \notin U$. As U is regular open, there is some $\tilde{y} \geqslant f(x)$, such that for all $\tilde{z} \geqslant \tilde{y}$, $\tilde{z} \notin U$. Then, as f is a UV map, by condition 2, we have some $y \in X$ such that $x \leqslant y$ and $f(y) = \tilde{y}$. Now let $z \geqslant y$. Then $f(z) \geqslant f(y) = \tilde{y}$, so $f(z) \notin U$, and $z \notin f^{-1}[U]$. So $f^{-1}[U]$ is regular open. Since f is a spectral map, $f^{-1}[U] \in \mathsf{CO}\mathcal{RO}(X)$.

Let $x \in f^{-1}[\mathsf{int}_{\leqslant}(Y \setminus U)]$. So $f(x)$ is an element of an upset contained in $Y \setminus U$, and in particular, $\uparrow f(x) \subseteq Y \setminus U$. To show $x \in \mathsf{int}_{\leqslant}(X \setminus f^{-1}[U])$, it is enough to show that $\uparrow x \subseteq X \setminus f^{-1}[U]$. So let $y \geqslant x$. Then $f(y) \geqslant f(x)$, so $f(y) \in Y \setminus U$, and $y \in X \setminus f^{-1}[U]$.

Now let $x \in \mathsf{int}_{\leqslant}(X \setminus f^{-1}[U])$. So $\uparrow x \subseteq X \setminus f^{-1}[U]$. To show $x \in f^{-1}[\mathsf{int}_{\leqslant}(Y \setminus U)]$, it is enough show $\uparrow f(x) \subseteq Y \setminus U$. So let $f(x) \leqslant \tilde{y}$. Then as f is a UV map, there is $y \in X$ with $x \leqslant y$ and $f(y) = \tilde{y}$. $y \in \uparrow x$, so $y \in X \setminus f^{-1}[U]$, and $\tilde{y} = f(y) \in Y \setminus U$, and we are done. ■

4.2 Duality

Having defined the morphisms and proved some useful results about them, we are now in position to complete the duality.

Definition 4.7. Let UV be the category of UV spaces with UV maps, as defined in the previous section[1].

Define functors

- $\alpha\colon \mathsf{BA} \to \mathsf{UV}$ where:
  - $\alpha(B) = UV(B)$;
  - for a homomorphism $h\colon A \to B$, let $h_+\colon UV(B) \to UV(A)$, sending $F \mapsto h^{-1}[F]$. Let $\alpha(h) = h_+$.
- $\beta\colon \mathsf{UV} \to \mathsf{BA}$ where:
  - $\beta(X) = \mathsf{CO}\mathcal{RO}(X)$;
  - for a UV map $f\colon X \to Y$, let $f^+\colon \mathsf{CO}\mathcal{RO}(Y) \to \mathsf{CO}\mathcal{RO}(X)$, sending $U \mapsto f^{-1}[U]$. Let $\beta(f) = f^+$.

Proposition 4.8. *α and β define contravariant functors.*

Proof. We need to show that $\alpha(h)$ is indeed a UV map, $\beta(f)$ is a homomorphism, and that both functors respect composition and identities.

[1] Checking this is actually a category requires checking that the composition of UV maps is a UV map, and that the identity map is a UV map; but this is obvious. And composition of functions is always associative.

$\alpha(h) = h_+$ is a UV map by Propositions 4.3 and 4.4, and $\beta(f) = f^+$ is a homomorphism by Proposition 4.6.

For any maps (not necessarily UV maps or homomorphisms) $f\colon X \to Y$ and $g\colon Y \to Z$, and a subset $U \subseteq Z$, $(g \circ f)^{-1}[U] = f^{-1}[g^{-1}[U]]$. Then $\alpha(h_1 \circ h_2) = \alpha(h_2) \circ \alpha(h_1)$, and similarly for β.

Also if h and f are the respective identity maps, then so are h_+ and f^+, and hence α and β are contravariant functors. ∎

Definition 4.9. Define natural transformations

- $\eta\colon \mathrm{id}_{\mathsf{BA}} \Rightarrow \beta\alpha$ where $\eta_B\colon B \to \beta\alpha(B) = \mathsf{CO}\mathcal{RO}(UV(B))$ is the map ϕ from Theorem 3.9.
- $\varepsilon\colon \alpha\beta \Rightarrow \mathrm{id}_{\mathsf{UV}}$ where $\varepsilon_X\colon UV(\mathsf{CO}\mathcal{RO}(X)) = \alpha\beta(X) \to X$ is the map θ^{-1} from Theorem 3.13.

Theorem 4.10. *η and ε are the unit and counit of a dual equivalence between* BA *and* UV.

Proof. This follows once we show the commutativity of the following two diagrams:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\
\mathsf{CO}\mathcal{RO}(UV(A)) \arrow[r, "\beta\alpha(h)"] & \mathsf{CO}\mathcal{RO}(UV(B))
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
UV(\mathsf{CO}\mathcal{RO}(X)) \arrow[u, "\varepsilon_X"] \arrow[r, "\alpha\beta(f)"] & UV(\mathsf{CO}\mathcal{RO}(Y)) \arrow[u, "\varepsilon_Y"]
\end{tikzcd}$$

Let $h\colon A \to B$ be a homomorphism, $a \in A$. $\beta\alpha(h)(\hat{a}) = (\alpha(h))^{-1}[\hat{a}]$, and so we show

$$(\alpha(h))^{-1}[\hat{a}] = \widehat{h(a)} \tag{$\star$}$$

Let F be a proper filter containing $h(a)$. $\alpha(h)(F) = h^{-1}[F]$ is clearly a proper filter containing a, giving the right-to-left direction of $(\star)$. Now let $F \in (\alpha(h))^{-1}[\hat{a}]$, so that $\alpha(h)(F) \in \hat{a}$, that is, $h^{-1}[F] \in \hat{a}$. So $h^{-1}[F]$ is a proper filter containing a. Then F is a proper filter containing $h(a)$, giving the left-to-right direction of $(\star)$. This proves commutativity of the first diagram.

Now let $f\colon X \to Y$ be a UV map and let $x \in X$. To show commutativity of the second diagram, we show $\alpha\beta(f)(\mathsf{CO}\mathcal{RO}(x)) = \mathsf{CO}\mathcal{RO}(f(x))$. $\alpha\beta(f)(\mathsf{CO}\mathcal{RO}(x)) = (\beta(f))^{-1}[\mathsf{CO}\mathcal{RO}(x)]$, so we show

$$(\beta(f))^{-1}[\mathsf{CO}\mathcal{RO}(x)] = \mathsf{CO}\mathcal{RO}(f(x)) \tag{$\dagger$}$$

Let $U \in \mathsf{CO}\mathcal{RO}(Y)$ contain $f(x)$. Then $\beta(f)[U] = f^{-1}[U]$ is in $\mathsf{CO}\mathcal{RO}(X)$, and contains x. This gives the right-to-left direction of (†). Now let $U \in (\beta(f))^{-1}[\mathsf{CO}\mathcal{RO}(x)]$. Then $\beta(f)[U] = f^{-1}[U] \in \mathsf{CO}\mathcal{RO}(x)$. $\beta(f)$ is a map $\mathsf{CO}\mathcal{RO}(Y) \to \mathsf{CO}\mathcal{RO}(X)$, so $U \in \mathsf{CO}\mathcal{RO}(Y)$, and since $f^{-1}[U] \in \mathsf{CO}\mathcal{RO}(x)$, $f(x) \in U$, and so $U \in \mathsf{CO}\mathcal{RO}(f(x))$. This gives the left-to-right direction, and hence the commutativity of the second diagram. ■

Chapter 5

Applications

5.1 Relationship between $UV(B)$ and the Stone dual of B

We now wish to examine how this UV dual of a Boolean algebra relates to the Stone dual. For this section, we will assume BPI. This section is based on Propositions 3.10 and 10.1 in [3].

Definition 5.1. Let X be a Stone space (Hausdorff, compact, with a basis of clopen sets). Let $\mathsf{F}(X)$ be the collection of non-empty closed subsets of X. For $U \in \mathsf{Clop}(X)$, let $\Box U = \{F \in \mathsf{F}(X) \mid F \subseteq U\}$. Note that $\Box U \cap \Box V = \Box(U \cap V)$. Define $\mathcal{UV}(X)$ to be the space $\mathsf{F}(X)$ with topology generated by $\{\Box U \mid U \in \mathsf{Clop}(X)\}$.

Proposition 5.2. *Let X be a Stone space. Then $\mathcal{UV}(X)$ is homeomorphic to $UV(\mathsf{Clop}(X))$.*

Proof. Let $f\colon \mathcal{UV}(X) \to UV(\mathsf{Clop}(X))$ send $C \mapsto \{U \in \mathsf{Clop}(X) \mid C \subseteq U\}$. Since C is non-empty, $f(C)$ is a proper filter in $\mathsf{Clop}(X)$.

Injective. Let $C \neq D$ be non-empty closed subsets of X, and without loss of generality, say $x \in C \setminus D$. D is closed so compact, and so we can find[1] disjoint open sets U_1, U_2 such that $D \subseteq U_1$ and $x \in U_2$. X has a basis of clopen sets, so write U_1 as a union of clopen sets. Then as D is compact, it has a finite subcover, and finite unions of clopen sets are clopen. So there is a clopen U such that $D \subseteq U \subseteq U_1$ and $x \in U_2$. Then $D \subseteq U$ and $x \notin U$, and so $C \nsubseteq U$. Then $f(C) \neq f(D)$, and f is injective.

Surjective. Let F be a proper filter in $\mathsf{Clop}(X)$. F has the finite intersection property, so by compactness, $\bigcap F$ is non-empty and an intersection of closed sets so is closed. We now show $f(\bigcap F) = F$. Clearly $F \subseteq f(\bigcap F)$. Let $U \supseteq \bigcap F$ be clopen. Then $X \setminus U \subseteq \bigcup\{X \setminus V \mid V \in F\}$. U is clopen so $X \setminus U$ is compact and so $X \setminus U \subseteq \bigcup\{X \setminus V \mid V \in F_0\}$ for some finite $F_0 \subseteq F$. Then $\bigcap F_0 \subseteq U$, and as F_0 is finite, $U \in F$.

[1] This result does not require any choice: consider the open cover of D, $\{U \mid U \text{ open}, U \cap D \neq \emptyset$, and there exists V open such that $x \in V, V \cap U = \emptyset\}$, let U_1 be the union of the finite subcover, and U_2 the intersection of the sets V corresponding to the finite subcover. This idea is taken from Asaf Karagila on Mathematics Stack Exchange [1].

Continuous. Let $\hat{U}$ be a basic open set in $UV(\mathsf{Clop}(X))$, so $U \in \mathsf{Clop}(X)$. Then

$$\begin{aligned} f^{-1}[\hat{U}] &= \{C \in \mathsf{F}(X) \mid f(C) \in \hat{U}\} \\ &= \{C \in \mathsf{F}(X) \mid U \in f(C)\} \\ &= \{C \in \mathsf{F}(X) \mid C \subseteq U\} \\ &= \Box U \end{aligned}$$

Continuous inverse. Now let $\Box U$ be a basic open set in $\mathcal{UV}(X)$. Then

$$\begin{aligned} f[\Box U] &= \{f(C) \mid C \in \Box U\} \\ &= \{f(C) \mid C \in \mathsf{F}(X) \text{ and } C \subseteq U\} \\ &= \{f(C) \mid U \in f(C)\} \\ &= \hat{U} \end{aligned}$$

The last equality follows since we have already shown the proper filters on $\mathsf{Clop}(X)$ are precisely the sets $f(C)$. ■

This proposition gives us a way of moving from the Stone dual of a Boolean algebra B to its choice-free dual. Let $S(B)$ denote the Stone dual of B. Then $\mathcal{UV}(S(B)) \cong UV(\mathsf{Clop}(S(B))) \cong UV(B)$. Note that this last point requires BPI since we use $B \cong \mathsf{Clop}(S(B))$, but the proof of the proposition does not use choice.

Now let B be a Boolean algebra, and $Y \subseteq UV(B)$ the subspace consisting of ultrafilters on B, with the subspace topology. Then

$$\begin{aligned} U \text{ is open in } Y &\Leftrightarrow U = V \cap Y \text{ for some } V \text{ open in } UV(B) \\ &\Leftrightarrow U = \left(\bigcup_{i \in I} \hat{a}_i\right) \cap Y \text{ for some } a_i \in B \\ &\Leftrightarrow U = \bigcup_{i \in I} (\hat{a}_i \cap Y) \text{ for some } a_i \in B \\ &\Leftrightarrow U \text{ is open in } S(B) \end{aligned}$$

The last equivalence is true since $\hat{a} \cap Y$ are the ultrafilters containing a, so these are the basic open sets in $S(B)$. Then $Y \cong S(B)$.

5.2 The dual space of the finite-cofinite algebra on $\mathbb{N}$

The finite-cofinite algebra on $\mathbb{N}$ is denoted $\mathrm{FC}(\mathbb{N})$, and consists of the finite and cofinite subsets of $\mathbb{N}$, with $\wedge, \vee, '$ given by intersection, union, and set-theoretic complement. To construct the dual space, we need to find all proper filters of $\mathrm{FC}(\mathbb{N})$. For each non-empty $A \in \mathrm{FC}(\mathbb{N})$, we have the proper filter $\uparrow A$, consisting of the sets containing A, that is, the principal filter generated by A.

Claim 1. If a proper filter contains a finite set, it is principal.

Proof. Suppose F is a proper filter in $\mathrm{FC}(\mathbb{N})$. If F contains a finite set, let $A \in F$ be a finite set of minimal cardinality (necessarily non-zero as F is proper). There is a unique such A since if there are two $A, B \in F$ such that A and B have the same smallest cardinality, then $A \cap B$ has smaller cardinality unless $A = B$. Then $F = {\uparrow}A$. ■

We will call a set $A \in \mathrm{FC}(\mathbb{N})$ *n-cofinite* if $\mathbb{N} \setminus A$ has size n. So a set is cofinite if and only if it is n-cofinite for some n.

Claim 2. Suppose F is a proper filter containing only cofinite sets. Suppose also that there is a minimal N such that every element of F is n-cofinite for some $n \leqslant N$. Then F is principal.

Proof. By minimality of N, there is at least one N-cofinite set. Suppose there are two: A and B. But then $A \cap B$ is at least $(N+1)$-cofinite unless one of A and B is contained in the other. But this would contradict both being N-cofinite unless $A = B$. So there is exactly one N-cofinite set, A, in F. It is then clear that $F = {\uparrow}A$. ■

We now consider the case when F is a proper filter containing only cofinite sets, but contains n-cofinite sets for arbitrarily large n. Note that if $A \in F$ is n-cofinite, then since F is a filter, $B \in F$ for every $B \supseteq A$. So F contains m-cofinite sets for every $m \leqslant n$. So it is equivalent to require F to contain n-cofinite sets for every n.

Claim 3. Let $\mathcal{F}$ be the set of proper filters containing only cofinite sets, and n-cofinite sets for every n. Then every $F \in \mathcal{F}$ is non-principal. Let $\mathcal{M}$ be the set of non-cofinite subsets of $\mathbb{N}$. Then there is a bijection:

$$\begin{aligned} \mathcal{F} &\leftrightarrow \mathcal{M} \\ F &\mapsto \bigcap F \\ F_M &\leftarrow\!\shortmid M \end{aligned}$$

where F_M is the proper filter containing all cofinite sets containing M.

Proof. If F was principal, then $F = {\uparrow}A$ for some n-cofinite A, but then F cannot contain any m-cofinite sets for $m > n$, so F is not principal. $\bigcap F$ is then a non-cofinite subset of $\mathbb{N}$.

Now let M be some non-cofinite subset of $\mathbb{N}$, and let F_M be the set of all cofinite sets containing M. Then clearly F_M is a non-principal proper filter, containing n-cofinite sets for every n.

It is clear that these operations are inverse to each other. ■

Then all three claims together prove that there is a bijection[2]

$$\begin{aligned} (\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}) \sqcup \mathcal{P}_f(\mathbb{N}) &\to \mathrm{PropFilt}(\mathrm{FC}(\mathbb{N})) \\ (M, 0) &\mapsto \begin{cases} {\uparrow}M & \text{if } M \text{ is finite or cofinite,} \\ F_M & \text{otherwise,} \end{cases} \\ (M, 1) &\mapsto F_M \end{aligned}$$

[2] $X \sqcup Y$ means the disjoint union, here given explicitly as $(X \times \{0\}) \cup (Y \times \{1\})$, and $\mathcal{P}_f(X)$ is the set of finite subsets of X.

Now we need to explicitly find the open sets $\hat{A}$ for each $A \in \mathrm{FC}(\mathbb{N})$. So let $A \in \mathrm{FC}(\mathbb{N})$ be finite. $\hat{A}$ consists of the proper filters containing A. These are the $\uparrow B$ for $B \subseteq A$. Now let $A \in \mathrm{FC}(\mathbb{N})$ be cofinite. Clearly if $B \in \mathrm{FC}(\mathbb{N})$ and $B \subseteq A$ then $A \in \uparrow B$.

Claim 4. For $A \in \mathrm{FC}(\mathbb{N})$ finite, $\hat{A} = \{\uparrow B \mid \emptyset \neq B \subseteq A\}$, and for $A \in \mathrm{FC}(\mathbb{N})$ cofinite, $\hat{A} = \{\uparrow B \mid B \subseteq A \text{ and } B \text{ is cofinite}\} \cup \{F_B \mid B \subseteq A \text{ and } B \text{ is not cofinite}\}$.

Proof. We have already shown that the only proper filters containing a finite set A are the principal ones $\uparrow B$ generated by a finite set. But $A \in \uparrow B \Leftrightarrow B \subseteq A$.

Now for $A \in \mathrm{FC}(\mathbb{N})$ cofinite, the only proper filters that may contain A are the $\uparrow B$ for B cofinite, or the F_B for B not cofinite. Similarly as before, for B cofinite, $B \subseteq A \Leftrightarrow A \in \uparrow B$, and for B not cofinite, $B \subseteq A \Leftrightarrow A \in F_B$. ■

We will translate these open sets on PropFilt(FC($\mathbb{N}$)) to open sets on $X = (\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}) \sqcup \mathcal{P}_f(\mathbb{N})$, to give a nicer description of the dual space, absent of any references to the original Boolean algebra, FC($\mathbb{N}$).

We will continue to use the technical definition of the disjoint union $X = (\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}) \sqcup \mathcal{P}_f(\mathbb{N})$. Corresponding to the open sets $\hat{A}$ where A is finite, we have open sets on X of the form $\{(B, 0) \mid \emptyset \neq B \subseteq A\}$. Corresponding to the open sets $\hat{A}$ where A is cofinite, we have open sets on X of the form $\{(B, 0) \mid B \subseteq A \text{ and } B \text{ is cofinite}\} \cup \{(B, 0) \mid B \subseteq A \text{ and } B \text{ is neither finite nor cofinite}\} \cup \{(B, 1) \mid B \subseteq A \text{ and } B \text{ is finite}\}$.

Let

$$X_A = \begin{cases} \{(B, 0) \mid B \subseteq A \text{ and } B \neq \emptyset\} & \text{if } A \text{ is finite,} \\ \begin{array}{l}\{(B, 0) \mid B \subseteq A \text{ and } B \text{ is not finite}\} \\ \quad \cup \{(B, 1) \mid B \subseteq A \text{ and } B \text{ is finite}\}\end{array} & \text{if } A \text{ is cofinite.} \end{cases}$$

Then $(X, \mathcal{T})$ is the dual space of FC($\mathbb{N}$), where $\mathcal{T}$ is the topology generated by the sets X_A.

5.3 Complete Boolean algebras

We now use the choice-free duality to prove a result about complete Boolean algebras. We will not give proofs for the propositions that lead up to the final result; they can all be found in [3, Sections 8, 9]. To begin with, we first characterise the UV-duals of complete Boolean algebras.

Definition 5.3. A UV space X is *complete* if $\mathsf{int}(\mathsf{cl}(U)) \in \mathsf{CO}\mathcal{RO}(X)$ for every open U. Note that int and cl are taken with respect to the spectral topology, not the upset topology.

Proposition 5.4. *Let B be a Boolean algebra and $X = UV(B)$. Then B is complete if and only if X is complete.*

Lemma 5.5. *If X is a complete UV space and $U \in \mathsf{CO}\mathcal{RO}(X)$, then U with the subspace topology is a complete UV space.*

Definition 5.6. An *isolated point* of a space X is $x \in X$ such that $\{x\}$ is open. Let X_{iso} be the set of isolated points of X.

It can be shown that $a \in B$ is an atom if and only if $\uparrow a \in UV(B)$ is an isolated point. Then B is atomless if and only if $UV(B)$ has no isolated points.

Proposition 5.7. *Let X be the dual space of B. Then B is atomic if and only if $\mathsf{int}(\mathsf{cl}(X_{iso})) = X$ if and only if $\mathsf{cl}(X_{iso}) = X$.*

Definition 5.8. Let X, Y be disjoint UV spaces. $X \bigcirc Y$ is the space with points $X \cup Y \cup (X \times Y)$ and topology generated by sets of the form $U \cup V \cup (U \times V)$ for $U \in \mathsf{CO}\mathcal{RO}(X)$, $V \in \mathsf{CO}\mathcal{RO}(Y)$. Then $X \bigcirc Y$ is the coproduct in the category UV, so that $X \bigcirc Y$ is a UV space, $UV(A) \bigcirc UV(B) \cong UV(A \times B)$, and $\mathsf{CO}\mathcal{RO}(X \bigcirc Y) \cong \mathsf{CO}\mathcal{RO}(X) \times \mathsf{CO}\mathcal{RO}(Y)$. This is called the *UV sum*.

Lemma 5.9. *Let X be a UV space, and $U \in \mathsf{CO}\mathcal{RO}(X)$. Then X is homeomorphic to the UV sum of the subspaces induced by U and $\neg U$.*

Proposition 5.10. *Any complete Boolean algebra is isomorphic to the product of a complete, atomless BA and a complete, atomic BA.*

Proof. By duality, it is enough to show a complete UV space X is the UV sum of a complete UV space with no isolated points, and a complete UV space with its isolated points as a dense subset. X is complete and X_{iso} is open so let $U = \mathsf{int}(\mathsf{cl}(X_{\text{iso}})) \in \mathsf{CO}\mathcal{RO}(X)$. Consider U and $\neg U$ as UV subspaces of X. Then X is the UV sum of these subspaces, both of which are complete.

Claim 1. If $V \subseteq X$ is open then $V_{\text{iso}} \subseteq X_{\text{iso}}$.
Proof of claim. Let $\{x\}$ be open in V. Then $\{x\} = V \cap W$ for some W open in X. But $V \cap W$ is open in X as both V, W are. So $\{x\}$ is open in X.

So $U_{\text{iso}} \subseteq X_{\text{iso}}$ and $(\neg U)_{\text{iso}} \subseteq X_{\text{iso}}$.

Claim 2. Let $V \subseteq X$ be open. If $X_{\text{iso}} \subseteq V$ then $X_{\text{iso}} \subseteq V_{\text{iso}}$.
Proof of claim. Let $\{x\}$ be open in X. Then $x \in V$ so $\{x\} = \{x\} \cap V$. So $\{x\}$ is open in V, and $x \in V_{\text{iso}}$.

X_{iso} is open so $X_{\text{iso}} \subseteq U$. Then by Claim 2, $X_{\text{iso}} \subseteq U_{\text{iso}}$, and $X_{\text{iso}} = U_{\text{iso}}$. As $U \cap \neg U = \emptyset$, $(\neg U)_{\text{iso}} = \emptyset$. Then in U, $\mathsf{int}^U(\mathsf{cl}^U(U_{\text{iso}})) = U \cap \mathsf{int}(\mathsf{cl}(U_{\text{iso}})) = U \cap U = U$, and we are done. ∎

Chapter 6

Ortholattices

We now turn our attention to ortholattices: essentially Boolean algebras without distributivity. In [8], Goldblatt proved a dual equivalence of ortholattices, taking the dual space to be the space of proper filters with basis $\hat{a}$ and $X \setminus \hat{a}$. However, this requires choice. In fact, if the basis is taken to be just the sets $\hat{a}$, the use of choice can be avoided, with the construction similar to that for Boolean algebras discussed in Chapters 3 and 4. The construction of the duality is taken from [19].

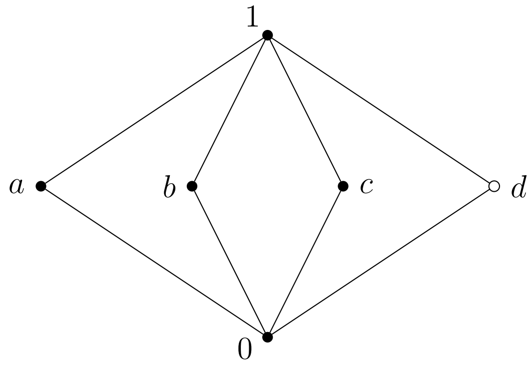
In [7], Goldblatt considered a form of non-classical logic: orthologic. This type of logic is similar to classical propositional logic, but lacks distributivity. It has been considered for quantum logic, since distributivity of 'and' and 'or' do not necessarily hold in quantum settings. An in depth discussion of quantum or orthologic, and where they might be used, is beyond the scope of this dissertation, but we will summarise some important results to provide motivation for studying ortholattices, once the necessary concepts are introduced. Rawling and Selesnick in [16] consider quantum logic in more detail, motivated by developments in quantum computation, and in [2], Bell extends Goldblatt's ideas to a predicate calculus for orthologic, though we focus only on the propositional case here.

6.1 Ortholattices

Definition 6.1. An *ortholattice* is a structure $\langle L, \vee, \wedge, 0, 1, {}' \rangle$, where $\langle L, \vee, \wedge, 0, 1 \rangle$ is a bounded lattice, and $'$ is a unary operation called *orthocomplementation* satisfying:

1. $a \wedge a' = 0$ and $a \vee a' = 1$;
2. $a'' = a$;
3. $a \leqslant b \Rightarrow b' \leqslant a'$;
4. $a \wedge b = (a' \vee b')'$.

Example. The following lattice is denoted $\mathbf{M}_4$.



Now we define the complementation function on $\mathbf{M}_4$, though note that only 0 and 1 have unique complements:

- $a' = b$, $b' = a$
- $c' = d$, $d' = c$

It can be seen that $\mathbf{M}_4$ is an ortholattice. We know that $\mathbf{M}_4$ is not distributive since it contains $\mathbf{M}_3$, 'the diamond', as a sublattice, given by the filled in nodes in the diagram. This is known as the $\mathbf{M}_3$-$\mathbf{N}_5$ Theorem in [4, Section 4.10]. So $\mathbf{M}_4$ is not a Boolean algebra.

Definition 6.2. An *orthoframe* $(X, \perp)$ is a set X and a irreflexive, symmetric (orthogononality) binary relation $\perp$. For any subset $Y \subseteq X$, we define $Y^{\perp} = \{a \in X \mid a \perp b \text{ for all } b \in Y\}$. Y is $\perp$-regular if $Y = Y^{\perp\perp}$. This is also called an *orthogonality space* in [8].

Definition 6.3. Let L be an ortholattice. Let X_L be the set of proper filters on L, with topology generated by $\hat{a} = \{F \in X_L \mid a \in F\}$. Then define a binary relation $\perp_L$ on X_L by

$$F \perp_L G \Leftrightarrow \text{there is } a \in L \text{ with } a \in F \text{ and } a' \in G.$$

Proposition 6.4. *$(X_L, \perp_L)$ is an orthoframe.*

Proof. Irreflexive. If $F \perp_L F$, then there is $a \in L$ such that $a \in F$ and $a' \in F$. As F is a filter, by condition 1 of Definition 6.1, $a \wedge a' = 0 \in F$, contradicting F being a proper filter. So $F \not\perp_L F$.

Symmetric. Suppose $F \perp_L G$. Then there is $a \in L$ with $a \in F$ and $a' \in G$. By condition 2 of Definition 6.1, $a'' = a \in F$, so that there is $a' \in L$ with $a' \in G$, and $a'' \in F$, that is, $G \perp_L F$. ■

6.2 Orthologic

Having introduced the necessary concepts, we now are in position to discuss orthologic, its relation to ortholattices, and how orthoframes can be used for orthologic semantics. This section is based on [7].

The language we will use contains a collection of propositional variables, and the symbols $\neg$ and $\wedge$ for negation and conjunction. Well-formed formulae are constructed in the usual way, and we will just refer to them as formulae, the set of which is denoted Φ.

Definition 6.5. An orthologic K is a collection of ordered pairs of formulae $\langle A, B\rangle$ (we write $A \vdash_K B$ for $\langle A, B\rangle \in K$) that satisfy the following conditions:

1. $A \vdash_K A$
2. $A \wedge B \vdash_K A$
3. $A \wedge B \vdash_K B$
4. $A \vdash_K \neg\neg A$
5. $\neg\neg A \vdash_K A$
6. $A \wedge \neg A \vdash_K B$
7. if $A \vdash_K B$ and $B \vdash_K C$ then $A \vdash_K C$
8. if $A \vdash_K B$ and $A \vdash_K C$ then $A \vdash_K B \wedge C$
9. if $A \vdash_K B$ then $\neg B \vdash_K \neg A$

The intersection of any family of orthologics is also an orthologic, so we denote the smallest orthologic by O. We could also define O in the following way. Call an expression of the form $\alpha \rightharpoonup \beta$ a sequent. Then $A \vdash_O B$ if there is a finite list of sequents, the last of which is $A \rightharpoonup B$, where each one is an axiom from conditions 1-6, or can be inferred from previous sequents using the rules of inference from conditions 7-9 (replace $\vdash_K$ by $\rightharpoonup$ in 1-9). This has the added benefit of being similar to the standard definition of a deductive system in propositional calculus, but Goldblatt's definition above has the benefit of being more general. The construction is detailed in [16, Section 2] and [2, Section 1].

It is easy to show that the Lindenbaum algebra (that is, $\Phi/\sim_K$ where $A \sim_K B$ if and only if $A \vdash_K B$ and $B \vdash_K A$) of any orthologic is an ortholattice. Using this result, we can also prove an algebraic characterisation theorem for O:

Theorem 6.6. *$A \vdash_O B \Leftrightarrow v(A) \leqslant v(B)$ for all valuations v on all ortholattices, where a valuation v on an ortholattice L is a map $\Phi \to L$ with $v(A \wedge B) = v(A) \wedge v(B)$ and $v(\neg A) = v(A)^{\perp}$.*

Definition 6.7. Let K be an orthologic and Γ a non-empty set of formulae.

1. A formula A is *K-derivable from* Γ, $\Gamma \vdash_K A$, if there are $B_1, B_2, \ldots, B_n \in \Gamma$ such that $B_1 \wedge \cdots \wedge B_n \vdash_K A$.
2. Γ is *K-consistent* if there is a formula not K-derivable from Γ.
3. Γ is *K-full* if it is K-consistent and closed under conjunction and K-derivability.

Remark. K-full sets of formulae correspond to proper filters of the Lindenbaum algebra of K.

One interesting aspect of orthologic is the proof of its version of Lindenbaum's lemma, [7, Theorem 1.4]. For classical propositional calculus, the proof of this requires some form of choice or an enumeration of all formulae, but for orthologic, the proof is quite direct.

Definition 6.8. $\mathcal{M} = (X, \perp, V)$ is an *orthomodel* on the orthoframe $(X, \perp)$ if V is a function from the set of propositional variables to the set of $\perp$-regular subsets of X. We can then define the truth of a formula A at x in $\mathcal{M}$ recursively:

1. $\mathcal{M} \vDash_x p$ for a propositional variable p if and only if $x \in V(p)$.
2. $\mathcal{M} \vDash_x A \wedge B$ if and only if $\mathcal{M} \vDash_x A$ and $\mathcal{M} \vDash_x B$.
3. $\mathcal{M} \vDash_x \neg A$ if and only if for any y, $\mathcal{M} \vDash_y A \Rightarrow x \perp y$.

Writing $\|A\|^{\mathcal{M}} = \{x \in X \mid \mathcal{M} \vdash_x A\}$, these are equivalent to:

1. $\|p\|^{\mathcal{M}} = V(p)$;
2. $\|A \wedge B\|^{\mathcal{M}} = \|A\|^{\mathcal{M}} \cap \|B\|^{\mathcal{M}}$;
3. $\|\neg A\|^{\mathcal{M}} = (\|A\|^{\mathcal{M}})^{\perp}$.

Let Γ be a non-empty set of formulae. Then:

1. Γ *implies* A at x in $\mathcal{M}$, $\mathcal{M} \colon \Gamma \vDash_x A$, if and only if $\mathcal{M} \vDash_x B \ \forall B \in \Gamma \Rightarrow \mathcal{M} \vDash_x A$.
2. Γ $\mathcal{M}$-*implies* A, $\mathcal{M} \colon \Gamma \vDash A$, if and only if Γ implies A at all $x \in \mathcal{M}$.
3. if $\mathcal{F}$ is a frame, Γ $\mathcal{F}$-*implies* A, $\mathcal{F} \colon \Gamma \vDash A$, if and only if $\mathcal{M} \colon \Gamma \vDash A$ for all models $\mathcal{M}$ on $\mathcal{F}$.
4. If $\mathcal{C}$ is a class of frames, Γ $\mathcal{C}$-*implies* A, $\mathcal{C} \colon \Gamma \vDash A$, if and only if $\mathcal{F} \colon \Gamma \vDash A$ for all frames $\mathcal{F} \in \mathcal{C}$.

To prove soundness and completeness for O, we need to construct the canonical orthomodel of an orthologic:

Definition 6.9. Let K be an orthologic. The *canonical orthomodel* for K is $\mathcal{M}_K = (X_K, \perp_K, V_K)$ where:

1. X_K is the set of all K-full sets of formulae.
2. $x \perp_K y$ if and only if there is a formula A such that $A \in x$ and $\neg A \in y$.
3. $V_K(p) = \{x \in X_K \mid p \in x\}$.

Note that the canonical model corresponds precisely with Definition 6.3.

Theorem 6.10. *Soundness and completeness for O*

$$\Gamma \vdash_O A \Leftrightarrow \theta \colon \Gamma \vDash A.$$

where θ is the class of all orthoframes.

6.3 Representation of ortholattices

We now return to proving a duality for ortholattices. First we prove some useful lemmas about $\perp$-regular subsets that lead to an important proposition.

Lemma 6.11. *Let $(X, \perp)$ be an orthoframe, and $A \subseteq X$ be any subset of X. Then $A \subseteq A^{\perp\perp}$.*

Proof. Let $a \in A$. To show $a \in A^{\perp\perp}$, we need to show $a \perp b$ for all $b \in A^{\perp}$. So let $b \in A^{\perp}$. Then $b \perp c$ for all $c \in A$. In particular, $b \perp a$, and by the symmetry of $\perp$, $a \perp b$, which proves the lemma. ■

Lemma 6.12. *Let $(X, \perp)$ be an orthoframe, and $A \subseteq B$ be two subsets of X. Then $B^{\perp} \subseteq A^{\perp}$.*

Proof. Let $b \in B^{\perp}$. We need to show $b \perp a$ for all $a \in A$. So let $a \in A$. Then as $A \subseteq B$, $a \in B$, so $b \perp a$, and we are done. ■

Proposition 6.13. *Let $(X, \perp)$ be an orthoframe. Then the collection of $\perp$-regular subsets of X is a complete ortholattice, ordered by inclusion, with $\wedge$ as set intersection, and complementation $'$ as $^{\perp}$. This ortholattice is denoted $R(X)$.*

Proof. Let $Y_i \subseteq X$ be $\perp$-regular for $i \in I$. First we show $\bigcap_{i \in I} Y_i$ is $\perp$-regular. Lemma 6.11 shows that $\bigcap_{i \in I} Y_i \subseteq (\bigcap_{i \in I} Y_i)^{\perp\perp}$. Now let $a \notin \bigcap_{i \in I} Y_i$. So for some $j \in I$, $a \notin Y_j$. Since Y_j is $\perp$-regular, $a \notin Y_j^{\perp\perp}$, and there is some $b \in Y_j^{\perp}$ with $a \not\perp b$. As $b \in Y_j^{\perp}$, lemma 6.12 gives that $b \in (\bigcap_{i \in I} Y_i)^{\perp}$. So $a \notin (\bigcap_{i \in I} Y_i)^{\perp\perp}$.

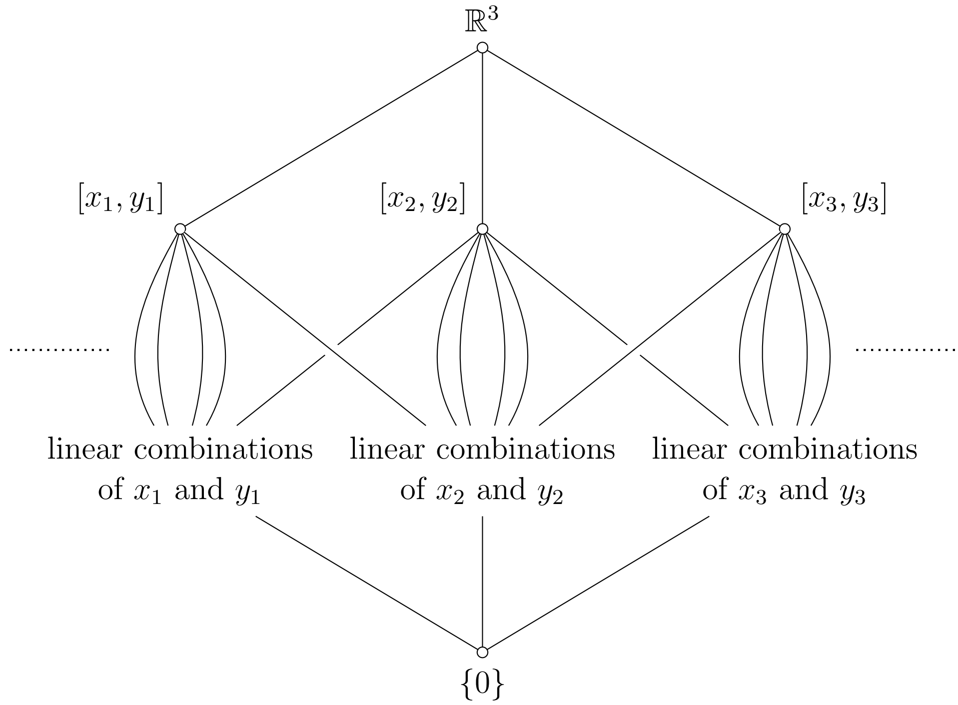
We have that $\emptyset^{\perp} = X$ vacuously, and $X^{\perp} = \emptyset$ by irreflexivity. So $\emptyset$ and X are $\perp$-regular, so are 0 and 1 in $R(X)$.

Let B be any $\perp$-regular subset of X containing $\bigcup_{i \in I} A_i$. Then two applications of Lemma 6.12 gives $(\bigcup_{i \in I} A_i)^{\perp\perp} \subseteq B^{\perp\perp}$. Since B is $\perp$-regular, $(\bigcup_{i \in I} A_i)^{\perp\perp} \subseteq B$. Then $\bigvee_{i \in I} A_i = (\bigcup_{i \in I} A_i)^{\perp\perp}$, since $(\bigcup_{i \in I} A_i)^{\perp\perp}$ is $\perp$-regular from Lemmas 6.11 and 6.12.

Now we need to show $\bigcap_{i \in I} A_i = (\bigvee_{i \in I} A_i^{\perp})^{\perp}$. Firstly, for each j, $A_j^{\perp} \subseteq \bigvee_{i \in I} A_i^{\perp}$, so that for each j, $(\bigvee_{i \in I} A_i^{\perp})^{\perp} \subseteq A_j$. Then $(\bigvee_{i \in I} A_i^{\perp})^{\perp} \subseteq \bigcap_{i \in I} A_i$. Now let $a \in \bigcap_{i \in I} A_i$, and $b \in \bigcup_{i \in I} A_i^{\perp}$. Then b is in some $A_j^{\perp}$, so $a \perp b$, and $a \in (\bigcup_{i \in I} A_i^{\perp})^{\perp} \subseteq (\bigcup_{i \in I} A_i^{\perp})^{\perp\perp\perp} = (\bigvee_{i \in I} A_i^{\perp})^{\perp}$. This proves the claim. ■

Example. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then for $x, y \in H \setminus \{0\}$, let $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. This is an orthogonality relation. A subset $Y \subseteq H \setminus \{0\}$ is $\perp$-regular if and only if $Y \cup \{0\}$ is a closed subspace of H. So we can think of the ortholattice of $\perp$-regular subsets as the closed subspaces of H.

In particular, for $H = \mathbb{R}^3$, the $\perp$-regular subsets are precisely all subspaces, since every subspace in a finite-dimensional space is closed. $R(H)$ has the following order with respect to inclusion:



where $[x, y]$ denotes the 2-dimensional subspace spanned by the non-zero vectors x and y. Also note that there is a 1-dimensional line lying in any two distinct planes, but this has been omitted from the diagram to make it less cluttered.

Theorem 6.14. *Let L be an ortholattice. Then there is an isomorphism $\phi\colon L \to \mathrm{COR}(X_L)$, where $\mathrm{COR}(X_L)$ is the ortholattice of compact open, $\perp$-regular subsets of X_L, a subortholattice of $R(X_L)$.*

Proof. For $a \in L$, let $\phi(a) = \hat{a}$.

Well-defined. First we need to show that $\hat{a}$ is compact and $\perp$-regular. So let $\hat{a} \subseteq \bigcup_{i \in I} \hat{b}_i$ for some $b \in L$. Consider the proper filter $\uparrow a \in \hat{a}$. Then $\uparrow a \in \hat{b}_j$ for some $j \in I$, so that $b_j \in \uparrow a$, and $a \leqslant b_j$. So any proper filter containing a also contains b_j, and $\hat{a} \subseteq \hat{b}_j$, and so $\hat{a}$ is compact.

Claim 1. $\hat{a}^\perp = \widehat{a'}$.
Proof of claim. The cases when a is 0 or 1 are obvious. Let $F \in \widehat{a'}$. Then for any proper filter G containing a, $F \perp_L G$, so that $F \in \hat{a}^\perp$.

Now let $F \in \hat{a}^\perp$. Let G be the filter $\uparrow a$. This is a proper filter since $a \neq 0$, so $F \perp G$. Then there is some $b \in L$ such that $b \in F$ and $b' \in G$. Then $b' \geqslant a$, so by condition 3 of Definition 6.1, $a' \geqslant b$. Since $b \in F$ and F is a filter, $a' \in F$, and $F \in \widehat{a'}$.

Then $\hat{a}^{\perp\perp} = \widehat{a'}^{\perp} = \widehat{a''} = \hat{a}$, and $\hat{a}$ is $\perp$-regular. Hence ϕ is well-defined.

Injective. If $\hat{a} = \hat{b}$, then $\uparrow a \in \hat{b}$ and $\uparrow b \in \hat{a}$. Then $a \leqslant b$ and $b \leqslant a$, and therefore $a = b$.

Homomorphism. We have that $\hat{a} \cap \hat{b} = \widehat{a \wedge b}$. $\cap$ is $\wedge$ on $R(X_L)$ and hence on $\mathrm{COR}(X_L)$. Claim 1 shows that ϕ preserves complements. Then by De Morgan's laws (condition 4 of Definition 6.1), ϕ preserves $\vee$ as well.

Surjectivity. Let $A \in \mathrm{COR}(X_L)$. As A is compact open, it is a finite union of basic open sets, so $A = \bigcup_{i=1}^n \hat{a}_i$. We wish to show that $A = \hat{b}$ for some $b \in L$.

Using the fact that A is $\perp$-regular,

$$\begin{aligned} A = A^{\perp\perp} &= (\bigcup_{i=1}^n \hat{a}_i)^{\perp\perp} \\ &= \bigvee_{i=1}^n \hat{a}_i \text{ by definition of } \vee. \\ &= \widehat{\bigvee_{i=1}^n a_i} \text{ since } \phi \text{ is a homomorphism.} \end{aligned}$$

So ϕ is surjective, and hence an isomorphism. ■

Theorem 6.15. *Let $(X, \perp)$ be an orthoframe with topology satisfying the following conditions:*

1. *T_0;*
2. *$\mathrm{COR}(X)$ is closed under $\cap$, $^\perp$, and forms a basis for the topology;*
3. *every proper filter of $\mathrm{COR}(X)$ is $\mathrm{COR}(x)$ for some $x \in X$, where $\mathrm{COR}(x)$ is the set of compact open, $\perp$-regular subsets of X containing x;*
4. *if $x \perp y$ then there is $U \in \mathrm{COR}(X)$ such that $x \in U$ and $y \in U^\perp$.*

Then $\rho(x) = \mathrm{COR}(x)$ is a homeomorphism $X \to X_{\mathrm{COR}(X)}$ that also preserves $\perp$.

Proof. Surjective. Condition 3 ensures ρ is surjective.

Injective. Suppose $x \neq y$. Then since X is T_0, there is an open set containing one but not the other. Since $\mathrm{COR}(X)$ is a basis, this open set can be taken to be $U \in \mathrm{COR}(X)$. So $\mathrm{COR}(x) \neq \mathrm{COR}(y)$.

Continuous. Let $\hat{U}$ be a basic open set in $X_{\mathrm{COR}(X)}$, where $U \in \mathrm{COR}(X)$. Then

$$\begin{aligned} \rho^{-1}[\hat{U}] &= \{x \in X \mid \rho(x) \in \hat{U}\} \\ &= \{x \in X \mid U \in \mathrm{COR}(x)\} \\ &= \{x \in X \mid x \in U\} \\ &= U \end{aligned}$$

Continuous inverse. Let $U \in \mathrm{COR}(X)$ be a basic open set in X. Then

$$\begin{aligned} \rho[U] &= \{\mathrm{COR}(x) \mid x \in U\} \\ &= \{\mathrm{COR}(x) \mid U \in \mathrm{COR}(x)\} \\ &= \{\mathrm{COR}(x) \mid \mathrm{COR}(x) \in \hat{U}\} \\ &= \hat{U} \end{aligned}$$

Preserves $\perp$.

$$\begin{aligned}\mathrm{COR}(x)\perp\mathrm{COR}(y) &\Leftrightarrow \text{there is some } U\in\mathrm{COR}(X) \text{ such that } U\in\mathrm{COR}(x)\\ &\ \text{and } U^{\perp}\in\mathrm{COR}(y)\\ &\Leftrightarrow \text{there is some } U\in\mathrm{COR}(X) \text{ such that } x\in U \text{ and } y\in U^{\perp}\\ &\Leftrightarrow x\perp y\end{aligned}$$

The last $\Rightarrow$ follows since if $y \in U^{\perp}$ then $y\perp z$ for every $z \in U$, and the last $\Leftarrow$ follows from condition 4. ■

6.4 Choice-free duality for ortholattices

We can extend these two theorems to a full dual equivalence between the category OL of ortholattices and the category UVO of orthoframes satisfying the conditions in Theorem 6.15. The morphisms of OL are the homomorphisms, and for UVO they are spectral maps $f\colon (X,\perp) \to (X',\perp')$ that are also p-morphisms:

- if $u\perp v$, then $f(u)\perp' f(v)$;
- if $s\perp' t$ and $s = f(u)$ for some $u \in X$, then there is some $v \in X$ such that $u\perp v$ and $t = f(v)$.

This is proved in [19, Proposition 2.2].

6.5 Specialising to Boolean algebras

We now investigate how this method relates to the method given in Chapters 3 and 4.

Lemma 6.16. *Let $A \subseteq UV(B)$. Then $A^{\perp}$ is an upset with respect to inclusion.*

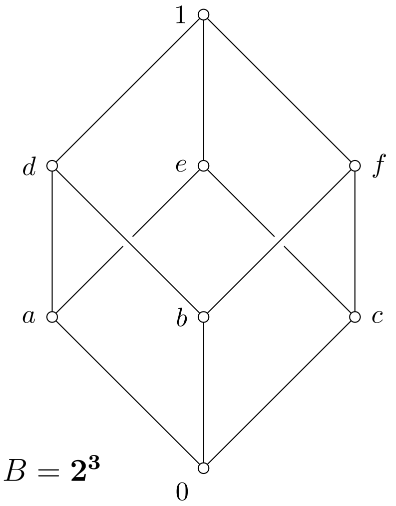
Proof. Let $F \in A^{\perp}$, and $G \supseteq F$. Now let $H \in A$. $F\perp H$, so there is $a \in F$ and $a' \in H$. So also, $a \in G$ and $a' \in H$. So $G \in A^{\perp}$. ■

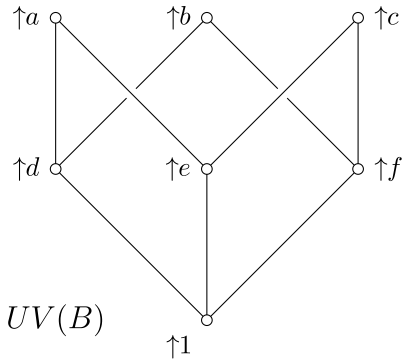
Lemma 6.17. *Let $A \subseteq UV(B)$ be an upset with respect to inclusion. Then $A^{\perp} = \mathsf{int}_{\subseteq}(UV(B) \setminus A)$.*

Proof. By lemma 6.16, $A^{\perp}$ is an upset, and is contained in $UV(B) \setminus A$. So $A^{\perp} \subseteq \mathsf{int}_{\subseteq}(UV(B) \setminus A)$.

Now let $F \in \mathsf{int}_{\subseteq}(UV(B) \setminus A)$, so that ${\uparrow}F \subseteq UV(B) \setminus A$. Let $G \in A$, and let H be the filter generated by $F \cup G$. If H is proper, then $H \in {\uparrow}F$ and $H \in A$, a contradiction. So H is improper, and there are $a \in F$, $b \in G$ such that $0 = a \wedge b$, as in Lemma B. Then, by Lemma A, $a' \geqslant b$, so $a' \in G$, and $F\perp G$. ■

Example. Let $B = \mathbf{2^3}$ and $A = \{{\uparrow}a, {\uparrow}f\} \subseteq UV(B)$.





For $F \perp \uparrow f$, we need $f' = a \in F$, since the only other element of $\uparrow f$ is 1. So $F = \uparrow a$. But then $\perp$ is irreflexive, so $A^{\perp} = \emptyset$. $\neg A$ is the largest upset contained in $UV(B) \setminus A$. From the diagram we can see that $\neg A = \{\uparrow b, \uparrow c\}$. So $^{\perp}$ and $\neg$ do not coincide in general on $UV(B)$.

Using the above two lemmas, we see that a subset of $UV(B)$ is $\perp$-regular if and only if it is regular open in the upset topology.

$$\begin{aligned} A = A^{\perp\perp} &\Leftrightarrow A = A^{\perp\perp} \text{ and } A \text{ is an upset, by Lemma 6.16} \\ &\Leftrightarrow A = A^{\perp\perp} \text{ and } A^{\perp\perp} = \mathsf{int}_{\subseteq}(UV(B) \setminus \mathsf{int}_{\subseteq}(UV(B) \setminus A)) \text{ by Lemma 6.17} \\ &\Leftrightarrow A = \mathsf{int}_{\subseteq}(\mathsf{cl}_{\subseteq}(A)) \end{aligned}$$

Word count: 7498.

Produced using TeXcount and includes all words in headers and footnotes, and excludes all mathematical symbols and formulae, the table of contents, the bibliography, and this sentence.

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