NEOSTABILITY TRANSFERS IN DERIVATION-LIKE THEORIES

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Abstract. Motivated by structural properties of differential field extensions, we introduce the notion of a theory T being derivation-like with respect to another model-complete theory T_0. We prove that when T admits a model-companion T_+, then several model-theoretic properties transfer from T_0 to T_+. These properties include completeness, quantifier-elimination, stability, simplicity, and NSOP_1. We also observe that, aside from the theory of differential fields, examples of derivation-like theories are plentiful.

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1. Introduction and preliminaries

Extending the argument of simplicity of the theory ACFA, in [4] Chatzidakis and Pillay studied the abstract condition of adding an automorphism to a first-order theory T_0 and proved that if such an expanded theory has a model-companion T_0A, then T_0A is simple whenever T_0 is stable (this stable-to-simple transfer result has been further generalised in [3]). In this paper we propose an abstract analogue of this where instead of adding an automorphism, we expand T_0 to a theory T that satisfies certain conditions which resemble structural properties of "derivations".

Recall that given a difference field (K,σ), the automorphism σ extends (not necessarily uniquely) to the separable closure K^{sep}. In the case of a differential field (K,δ) much more is true: the derivation extends *uniquely* to any separably algebraic extension. This is a crucial difference between the theories of difference fields and differential fields; for instance, it is one of the reasons why DCF_0 has quantifier elimination while ACFA does not. Another structural property of differential fields (or even differential rings) is that given two differential fields (K,δ_1) and (L,δ_2)

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with a common differential subfield (E, δ), the tensor product $K \otimes_E L$ has a *unique* derivation extending those on K and L (note that this property also holds for difference fields).

We extract the above two properties of differential field extensions to an abstract setup and define (in Definition 2.1) the notion of a theory T being derivation-like with respect to a complete and model-complete theory T_0 equipped with an invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$. The motivating example, of course, is that the theory of differential fields in characteristic zero DF_0 is derivation-like with respect to ACF_0 equipped with the algebraic-disjointness relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$. In §3, we provide several other instances of derivation-like theories; in particular, we note that the recently developed theory DCCM of compact complex manifolds with meromorphic vector fields, introduced by Moosa in [14], is derivation-like.

In §2, under the assumption that T has a model-companion T_+ (and some assumptions on $\mathop{\smile\!\!\!\!\!\!|}^{0}$), we prove that several model-theoretic properties transfer from T_0 to T_+. In particular, completeness and quantifier-elimination transfer, and we also observe that the model-theoretic dcl and acl have a natural description. In order to state the other neostability transfers, let us briefly recall what is meant by an independence relation in T_0. We somewhat follow the presentation of Adler [1].

Definition 1.1. A relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$ on triples of small subsets of a monster model $\mathcal{U}_0$ of T_0 is called an independence relation if it is invariant under automorphisms and satisfies the following eight properties:

(1) *normality:* $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B \implies A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} BC$;
(2) *monotonicity:* $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} BD \implies A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$;
(3) *base monotonicity:* $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} BD \implies A \mathop{\smile\!\!\!\!\!\!|}^{0}_{CD} B$;
(4) *transitivity:* $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$ and $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{B} D \implies A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} D$ for $C \subseteq B \subseteq D$;
(5) *symmetry:* $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B \implies B \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} A$;
(6) *full existence:* for any A, B, C there is $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$;
(7) *finite character:* if $A_0 \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$ for all finite $A_0 \subseteq A$ then $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$;
(8) *local character:* for any A there is a cardinal $\kappa = \kappa(A)$ such that for any B there is $C \subseteq B$ with $|C| < \kappa$ such that $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$.

There are other properties that are generally of interest:

- *existence:* for any A and C we have $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} C$;
- *extension:* if $A \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} B$ then for any D there is $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} BD$;
- *anti-reflexivity:* if $a \mathop{\smile\!\!\!\!\!\!|}^{0}_{C} a$ then $a \in \operatorname{acl}(C)$ (an independence relation is called strict if it satisfies anti-reflexivity);
- *chain local character:* for a a finite tuple and $\kappa > |T_0|$ a regular cardinal, for every continuous chain of models $(M_i)_{i<\kappa}$ with $|M_i| < \kappa$ there is $j < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!|}^{0}_{M_j} \cup_{i<\kappa} M_i$;
- *independence theorem over M:* if $A_1 \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A_2$, $a_1 \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A_1$, $a_2 \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A_2$, and $a_1 \equiv_M a_2$, then there is $a \models \operatorname{tp}(a_1/MA_1) \cup \operatorname{tp}(a_2/MA_2)$ with $a \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A_1 A_2$;
- *stationarity over M:* if $M \subseteq A$, $a \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A$, $b \mathop{\smile\!\!\!\!\!\!|}^{0}_{M} A$, and $a \equiv_M b$, then $a \equiv_A b$.

While most neostability properties have local combinatorial descriptions, the Kim–Pillay type theorems indicate semantic ways to describe such properties in terms of ternary relations. We summarise the developments in this direction that will be relevant for us.

Theorem 1.2.

(i) [10] *The theory T_0 is stable if and only it admits an independence relation $\mathop{\smile\!\!\!\!\!\!|}^0$ (i.e., satisfying (1)-(8) above) which satisfies stationarity over models. Moreover, in this case $\mathop{\smile\!\!\!\!\!\!|}^0$ coincides with forking independence.*

(ii) [11] *The theory T_0 is simple if and only it admits an independence relation $\mathop{\smile\!\!\!\!\!\!|}^0$ which satisfies the independence theorem over models. Moreover, in this case $\mathop{\smile\!\!\!\!\!\!|}^0$ coincides with forking independence.*

(iii) [5] *The theory T_0 is NSOP$_1$ if and only if it admits an invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}^0$ with chain local character and which over models satisfies monotonicity, transitivity, symmetry, finite character, existence, extension, and the independence theorem. Moreover, in this case $\mathop{\smile\!\!\!\!\!\!|}^0$ coincides with Kim-independence over models.*

(iv) [1] *Suppose T_0 eliminates imaginaries. Then, T_0 is rosy if and only if it admits a strict independence relation.*

By inducing (from $\mathop{\smile\!\!\!\!\!\!|}^0$) a natural ternary relation $\mathop{\smile\!\!\!\!\!\!|}^+$ on the model companion T_+ and using Theorem 1.2, in §2 we are able to prove that stability and simplicity transfer from T_0 to T_+. Furthermore, when T_0 is the theory of a very slim field and T is in addition derivation-like w.r.t. $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$, we also prove that NSOP$_1$ transfers to T_+. Finally, assuming that both T_0 and T_+ eliminate imaginaries, we obtain that Rosiness also transfers. Our method of proof relies on a detailed study on how the independence properties in Definition 1.1 transfer from T_0 to T_+. This is explicitly done in Theorems 2.11, 2.13, and 2.14.

Conventions. We assume that all our theories are closed under deductions.

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2. Main results

We fix the following data:

- $\mathcal{L}_0 \subseteq \mathcal{L}$ are two (first-order) languages, possibly multi-sorted,
- T_0 is a complete and model-complete $\mathcal{L}_0$-theory equipped with an automorphism invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}^0$, we denote by $\mathcal{U}_0$ a monster model of T_0 and, unless otherwise stated, acl_0 refers to model-theoretic algebraic closure taken with respect to the language $\mathcal{L}_0$ in $\mathcal{U}_0$, and
- T is an $\mathcal{L}$-theory such that $T_0^{\forall} \subseteq T$.

Definition 2.1. We say that T is derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!|}^0)$ if whenever $A, B, C \models T^{\forall}$, with C a common $\mathcal{L}$-substructure of A and B, are such that $A, B \leq_{\mathcal{L}_0} \mathcal{U}_0$, $\mathrm{acl}_0(C) \cap A = \mathrm{acl}_0(C) \cap B = C$, and $A \mathop{\smile\!\!\!\!\!\!|}^0_C B$, we have that

(i) there exists $M \models T$ such that $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $A, B \leq_{\mathcal{L}} M$, and

(ii) for any M as in (i) and any $\mathcal{L}_0$-structure D such that

$$\langle A, B\rangle_{\mathcal{L}_0} \leq_{\mathcal{L}_0} D \leq_{\mathcal{L}_0} \operatorname{acl}_0(A,B)\cap M,$$

we have that $D \leq_{\mathcal{L}} M$ and, moreover, this $\mathcal{L}$-structure on D is the unique one expanding its $\mathcal{L}_0$-structure, making it a model of $T^{\forall}$, and extending the $\mathcal{L}$-structures of A and B.

Note that part (i) of the definition is, in some sense, a strong form of independent amalgamation.

Remark 2.2. Suppose T is derivation-like with respect to $(T, \mathop{\smile\!\!\!\!\!\!|}\,^0)$ and $M \models T$ with $M \leq_{\mathcal{L}_0} \mathcal{U}_0$. We note that if $A \leq_{\mathcal{L}} M$ is such that

$$A \mathop{\smile\!\!\!\!\!\!|}^{0}_{A} A,$$

then $\operatorname{acl}_0(A)\cap M \leq_{\mathcal{L}} M$. Indeed, taking B and C equal to A, part (ii) of the definition yields

$$\operatorname{acl}_0(A)\cap M = \operatorname{acl}_0(A,B)\cap M \leq_{\mathcal{L}} M.$$

More generally, whenever D is an $\mathcal{L}_0$-structure such that $A \leq_{\mathcal{L}_0} D \leq_{\mathcal{L}_0} \operatorname{acl}_0(A)\cap M$, we then have that $D \leq_{\mathcal{L}} M$ and this $\mathcal{L}$-structure on D is the unique one expanding its $\mathcal{L}_0$-structure, making it a model of $T^{\forall}$, and extending the $\mathcal{L}$-structure of A.

The following assumptions will be in place throughout the rest of this section.

Assumption 2.3.

(i) From now on T is a derivation-like theory with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!|}\,^0)$.
(ii) We assume that T has a model companion T_+ and that $T \subseteq T_+$. We fix a monster model $\mathcal{U}_+$ of a completion of T_+. Since $T_0^{\forall} \subseteq T_+$, without loss of generality we may assume that $\mathcal{U}_+ \leq_{\mathcal{L}_0} \mathcal{U}_0$. acl_+ refers to model-theoretic algebraic closure taken in $\mathcal{U}_+$ (w.r.t. the language $\mathcal{L}$).
(iii) If $T_0 \not\subseteq T_+$, we further assume that T_0 has quantifier elimination.

Let $\mathcal{L}_0^*$ be some language expanding $\mathcal{L}_0$, and set $\mathcal{L}^* = \mathcal{L}\cup\mathcal{L}_0^*$. Let T_0^* be an expansion by definitions of T_0 to the language $\mathcal{L}_0^*$ (e.g., the Morleyisation of T_0). Also, expand T and T_+ to T^* and T_+^*, respectively, to the language $\mathcal{L}^*$ using the same definitions as for T_0^*.

Remark 2.4. The following can be readily checked:

(1) $(T_0^*)^{\forall} \subseteq T^*$;
(2) $\mathop{\smile\!\!\!\!\!\!|}\,^0$ is naturally an invariant ternary relation on $\mathcal{U}_0$ as a model of T_0^*;
(3) T^* is derivation-like with respect to $(T_0^*, \mathop{\smile\!\!\!\!\!\!|}\,^0)$;
(4) T_+^* is the model companion of T^* and $T^* \subseteq T_+^*$; and
(5) $\mathcal{U}_0$ and $\mathcal{U}_+$ remain monster models of T_0^* and T_+^*, respectively.

Lemma 2.5. *Assume $\mathop{\smile\!\!\!\!\!\!|}\,^0$ satisfies full existence. Let $A \leq_{\mathcal{L}} \mathcal{U}_+$. If $T_0^*\cup\operatorname{diag}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(A)$ is complete, then $T_+^*\cup\operatorname{diag}^{\mathcal{U}_+}_{\mathcal{L}^*}(A)$ is complete.*

Proof. Let $K \models T_+^*\cup\operatorname{diag}^{\mathcal{U}_+}_{\mathcal{L}^*}(A)$. We will show that $K \equiv_A \mathcal{U}_+$ as $\mathcal{L}^*$-structures. First note that $K \models (T_0^*)^{\forall}$, and hence it $\mathcal{L}_0^*$-embeds in some $K' \models T_0^*$. Now by completeness of $T_0^*\cup\operatorname{diag}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(A)$, K' $\mathcal{L}_0^*$-embeds inside $\mathcal{U}_0$ over A. Let L be an $\mathcal{L}^*$-elementary substructure of $\mathcal{U}_+$ containing A. Use full existence to find a copy of L

with $L' \mathop{\smile\!\!\!\!\!\!\;|}^0_A K$ and $\mathrm{tp}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(L'/A) = \mathrm{tp}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(L/A)$. This last fact means that L induces an isomorphic $\mathcal{L}^*$-structure on L'. The partial $\mathcal{L}_0^*$-elementary map $A \to A$ from K to L' extends to a partial $\mathcal{L}_0^*$-elementary map $\mathrm{acl}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(A) \cap K \to \mathrm{acl}^{\mathcal{U}_0}_{\mathcal{L}_0^*}(A) \cap L'$. By Remark 2.2, this map must be an $\mathcal{L}^*$-isomorphism (note that full existence yields $A \mathop{\smile\!\!\!\!\!\!\;|}^0_A A$). So we may assume that A is relatively $\mathrm{acl}_{\mathcal{L}^*}$-closed in K and L'.

Since T^* is derivation-like, there is some $M \models T^*$ such that $M \leq_{\mathcal{L}_0^*} \mathcal{U}_0$ and $K, L' \leq_{\mathcal{L}^*} M$. Since T_+^* is the model companion of T^*, there is some $N \models T_+^*$ extending M as an $\mathcal{L}^*$-structure. Now, T_+^* is model complete, so $L' \preceq N \succeq K$ as $\mathcal{L}^*$-structures. Finally $K \equiv_A N \equiv_A L' \equiv_A L \equiv_A \mathcal{U}_+$. $\square$

We collect some immediate corollaries.

Corollary 2.6. *Assume $\mathop{\smile\!\!\!\!\!\!\;|}^0$ satisfies full existence. Suppose T_0^* is the model companion of some inductive $\mathcal{L}_0^*$-theory S. Then*

(1) *$T_+ \cup S$ is the model companion of $T \cup S$;*
(2) *if T_0^* is the model completion of S, then $T_+ \cup S$ is the model completion of $T \cup S$; and*
(3) *if T_0^* has quantifier elimination, then T_+^* has quantifier elimination.*

Without loss of generality (due to Remark 2.4), by possibly passing to the Morleyisation of T_0, from this point on we **assume** that T_0 has quantifier elimination.

Lemma 2.7. *Assume $\mathop{\smile\!\!\!\!\!\!\;|}^0$ satisfies monotonicity, symmetry, full-existence, and anti-reflexivity. Then, for any $A \subset \mathcal{U}_+$, we have*

$$\mathrm{acl}_+(A) = \mathrm{acl}_0(\langle A\rangle_{\mathcal{L}}) \cap \mathcal{U}_+.$$

Proof. By full-existence, we have that $A \mathop{\smile\!\!\!\!\!\!\;|}^0_A A$, and so by Remark 2.2 we have that

$$F := \mathrm{acl}_0(\langle A\rangle_{\mathcal{L}}) \cap \mathcal{U}_+$$

is an $\mathcal{L}$-substructure of $\mathcal{U}_+$. As T_0 has q.e., we get $F \subseteq \mathrm{acl}_+(A)$. For the other containment, consider $a \in \mathrm{acl}_+(A)$. Let K be an elementary $\mathcal{L}$-substructure of $\mathcal{U}_+$ containing all (finitely-many) realisations of $\mathrm{tp}_+(a/F)$. By full-existence, there is an $\mathcal{L}_0$-substructure L of $\mathcal{U}_0$ such that $L \mathop{\smile\!\!\!\!\!\!\;|}^0_F K$ and $\mathrm{tp}_0(L/F) = \mathrm{tp}_0(K/F)$. The latter induces an $\mathcal{L}$-structure on L, via some $\sigma \in \mathrm{Aut}_{\mathcal{L}_0}(\mathcal{U}_0/F)$ with $L = \sigma(K)$, making $L \models T_+$ and an $\mathcal{L}$-extension of F. Since T is derivation-like, there is $M \models T$ with $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ such that K and L are $\mathcal{L}$-substructures of M. Let $N \models T_+$ be an $\mathcal{L}$-extension of M. Since T_+ is model-complete and $K \models T_+$ is a common substructure of N and $\mathcal{U}_+$, there is an elementary $\mathcal{L}$-embedding $\phi : N \to \mathcal{U}_+$ over K. Let $L' = \phi(L)$. We first note that

$$\mathrm{tp}_+^{\mathcal{U}_+}(a/F) = \mathrm{tp}_+^{K}(a/F) = \mathrm{tp}_+^{L}(\sigma(a)/F) = \mathrm{tp}_+^{N}(\sigma(a)/F) = \mathrm{tp}_+^{\mathcal{U}_+}(\phi(\sigma(a))/F)$$

and so $\phi(\sigma(a)) \in K$ (as K contains all realisations of $\mathrm{tp}_+^{\mathcal{U}_+}(a/F)$).

We now claim that $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(L'/K)$. First note that

$$\mathrm{qftp}_0^{\mathcal{U}_0}(L/K) = \mathrm{qftp}_0^{M}(L/K) = \mathrm{qftp}_0^{N}(L/K) = \mathrm{qftp}_0^{\mathcal{U}_+}(L'/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(L'/K).$$

Since T_0 admits q.e., it follows that $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(L'/K)$.

Now, by invariance, we have $L' \mathop{\smile\!\!\!\!\!\!\;|}^0_F K$. Then, monotonicity and symmetry imply that $\phi(\sigma(a)) \mathop{\smile\!\!\!\!\!\!\;|}^0_F \phi(\sigma(a))$. By anti-reflexivity, $\phi(\sigma(a)) \in \mathrm{acl}_0(F) \cap \mathcal{U}_+ = F$. Since $\phi \circ \sigma$ fixes F pointwise, we get that $a \in F$, as desired.

$\square$

We now observe that in derivation-like theories with $T_0 \subseteq T_+$ we have a natural description of dcl.

Lemma 2.8. *Assume $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies monotonicity, symmetry, full-existence, and anti-reflexivity. In addition, assume that $T_0 \subseteq T_+$. Then, for any $A \subset \mathcal{U}_+$, we have*

$$\mathrm{dcl}_+(A) = \mathrm{dcl}_0(\langle A \rangle_{\mathcal{L}}).$$

Proof. As T_0 has q.e., we have

$$\mathrm{dcl}_0(\langle A \rangle_{\mathcal{L}}) \subseteq \mathrm{dcl}_+(A).$$

For the other containment, let $a \in \mathrm{dcl}_+(A)$ and let σ be an $\mathcal{L}_0$-automorphism of $\mathcal{U}_0$ fixing $\langle A \rangle_{\mathcal{L}}$ pointwise. We aim to show that $\sigma(a) = a$. Note that the assumption $T_0 \subseteq T_+$ implies $\mathcal{U}_+ \preceq_{\mathcal{L}_0} \mathcal{U}_0$; this together with Lemma 2.7 yields

$$\sigma(\mathrm{dcl}_+(A)) \leq_{\mathcal{L}_0} \sigma(\mathrm{acl}_0(\langle A \rangle_{\mathcal{L}}) = \mathrm{acl}_0(\langle A \rangle_{\mathcal{L}}) \leq_{\mathcal{L}_0} \mathcal{U}_+.$$

Because T is derivation-like, by Remark 2.2, we have that $\sigma(\mathrm{dcl}_+(A)) \leq_{\mathcal{L}} \mathcal{U}_+$ and this is the unique $\mathcal{L}$-structure expanding its $\mathcal{L}_0$-structure, making it a model of $T^\forall$, and extending $\langle A \rangle_{\mathcal{L}}$. It follows from this and the fact that T_+ admits q.e. (by Corollary 2.6(3)), that σ restricted to $\mathrm{dcl}_+(A)$ is a partial $\mathcal{L}$-elementary map of $\mathcal{U}_+$. Thus, we may extend this restriction to an automorphism ρ of $\mathcal{U}_+$ (which fixes A). But then, as $a \in \mathrm{dcl}_+(A)$, we have that $a = \rho(a) = \sigma(a)$. $\square$

Remark 2.9. We note that in the proofs of Lemmas 2.5, 2.7, 2.8, condition (ii) of derivation-like was only used when $A = B = C$. Namely, we applied Remark 2.2. In other words, one could say that T is *almost* derivation-like if in condition (ii) we restrict only to $A = B = C$; and then all results stated so far hold when T is almost derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!|}^0)$.

Definition 2.10. Define the following relation on triples of small subsets of $\mathcal{U}_+$:

$$A \mathop{\smile\!\!\!\!\!\!|}^{+}_{C} B \iff \mathrm{acl}_+(AC) \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(BC).$$

The following provides a detailed description of how independence properties of $\mathop{\smile\!\!\!\!\!\!|}^0$ transfer to $\mathop{\smile\!\!\!\!\!\!|}^+$.

Theorem 2.11. (1) *$\mathop{\smile\!\!\!\!\!\!|}^+$ is invariant and normal;*

(2) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies any of monotonicity, symmetry, finite character, then so does $\mathop{\smile\!\!\!\!\!\!|}^+$;*
(3) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ is transitive and monotone, then $\mathop{\smile\!\!\!\!\!\!|}^+$ is transitive;*
(4) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies base monotonicity, finite character, and local character, then $\mathop{\smile\!\!\!\!\!\!|}^+$ has local character;*
(5) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies normality, monotonicity, base monotonicity, transitivity, symmetry, and full existence, then $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies base monotonicity;*
(6) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies monotonicity, symmetry, and full existence, then $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies full existence;*
(7) *if $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies monotonicity and extension and T_+ has quantifier elimination, then $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies extension.*

In addition, if $T_0 \subseteq T_+$, then (2), (3), and (7) *also hold for the corresponding properties stated over models.*

Proof. Invariance. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} B$ and $\mathrm{tp}_+(ABC) = \mathrm{tp}_+(A'B'C')$. Then

$$\mathrm{tp}_+(\mathrm{acl}_+(ABC)) = \mathrm{tp}_+(\mathrm{acl}_+(A'B'C')),$$

and similar arguments to the proofs above (using q.e. for T_0) show that

$$\mathrm{tp}_0(\mathrm{acl}_+(ABC)) = \mathrm{tp}_0(\mathrm{acl}_+(A'B'C')).$$

Invariance for $\mathop{\smile\!\!\!\!\!\mid}^{0}$ then means that $A' \mathop{\smile\!\!\!\!\!\mid}^{+}_{C'} B'$.

Normality is by definition (i.e. does not require normality of $\mathop{\smile\!\!\!\!\!\mid}^{0}$).

Monotonicity. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} B$ and $D \subseteq B$. Then $\mathrm{acl}_+(AC) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(BC)$. Also $\mathrm{acl}_+(DC) \subseteq \mathrm{acl}_+(BC)$, so by monotonicity for $\mathop{\smile\!\!\!\!\!\mid}^{0}$, we have that

$$\mathrm{acl}_+(AC) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(DC);$$

that is, $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} D$.

Transitivity follows from transitivity and monotonicity of $\mathop{\smile\!\!\!\!\!\mid}^{0}$.

Symmetry follows from symmetry of $\mathop{\smile\!\!\!\!\!\mid}^{0}$.

Finite character follows from finite character of $\mathop{\smile\!\!\!\!\!\mid}^{0}$ and the fact that acl_+ is finitary.

Local character. Precisely the same proof as in Theorem 2.1 of [3] applies here.

Base monotonicity. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} B$ and $C \subseteq D \subseteq B$. We may also assume that $A \supseteq C$ by normality. Then $\mathrm{acl}_+(A) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(B)$. By monotonicity, we have $\mathrm{acl}_+(A) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(C)} \mathrm{acl}_+(D)$. Since T is derivation-like,

$$\mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+ \leq_{\mathcal{L}} \mathcal{U}_+.$$

So $\langle AD\rangle_{\mathcal{L}} \subseteq \mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+$, and so by Lemma 2.7 we have

$$\mathrm{acl}_+(AD) = \mathrm{acl}_0(\langle AD\rangle_{\mathcal{L}}) \cap \mathcal{U}_+ \subseteq \mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \cap \mathcal{U}_+.$$

By base monotonicity and normality for $\mathop{\smile\!\!\!\!\!\mid}^{0}$, we get

$$\mathrm{acl}_+(A)\,\mathrm{acl}_+(D) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(D)} \mathrm{acl}_+(B).$$

By full existence, we get

$$\mathrm{acl}_0(\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(A)\,\mathrm{acl}_+(D)} \mathrm{acl}_+(B),$$

and by symmetry, transitivity, and monotonicity, $\mathrm{acl}_+(AD) \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(D)} \mathrm{acl}_+(B)$. That is, $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{D} B$.

For *full existence*, suppose a, A, B are given inside $\mathcal{U}_+$. We need to find $a' \in \mathcal{U}_+$ such that $\mathrm{tp}_+(a'/A) = \mathrm{tp}_+(a/A)$ and $a' \mathop{\smile\!\!\!\!\!\mid}^{+}_{A} B$. Let K be some small $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing a, A, B. Write $C = \mathrm{acl}_+(A)$. Use full existence for $\mathop{\smile\!\!\!\!\!\mid}^{0}$ to find $L \leq_{\mathcal{L}_0} \mathcal{U}_0$ with $L \mathop{\smile\!\!\!\!\!\mid}^{0}_{C} K$ and $\mathrm{tp}_0(L/C) = \mathrm{tp}_0(K/C)$. Let $\sigma \in \mathrm{Aut}(\mathcal{U}_0/C)$ be the $\mathcal{L}_0$-automorphism taking K to L. This automorphism then induces an $\mathcal{L}$-structure on L. Since T is derivation-like, there is some $M \models T$ such that $K, L \leq_{\mathcal{L}} M \leq_{\mathcal{L}_0} \mathcal{U}_0$. Since T_+ is the model companion of T, there is some $N \models T_+$

extending M. Let $\phi\colon N \to \mathcal{U}_+$ be the $\mathcal{L}$-elementary embedding of N inside $\mathcal{U}_+$ that fixes K. Then

$$\mathrm{tp}_+^{\mathcal{U}_+}(a/C) = \mathrm{tp}_+^{K}(a/C) = \mathrm{tp}_+^{L}(\sigma(a)/C) = \mathrm{tp}_+^{N}(\sigma(a)/C) = \mathrm{tp}_+^{\mathcal{U}_+}(\phi\sigma(a)/C).$$

We also have the following chain of equalities of quantifier-free $\mathcal{L}_0$-types:

$$\mathrm{qftp}_0^{\mathcal{U}_0}(L/K) = \mathrm{qftp}_0^{M}(L/K) = \mathrm{qftp}_0^{N}(L/K) = \mathrm{qftp}_0^{\mathcal{U}_+}(\phi(L)/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(\phi(L)/K).$$

As T_0 has q.e., this yields $\mathrm{tp}_0(L/K) = \mathrm{tp}_0(\phi(L)/K)$. Invariance then gives $\phi(L) \mathop{\smile\!\!\!\!\!\mid}^{0}_{C} K$, and monotonicity gives $\mathrm{acl}_+(C, \phi\sigma(a)) \mathop{\smile\!\!\!\!\!\mid}^{0}_{C} \mathrm{acl}_+(AB)$; that is, $\phi\sigma(a) \mathop{\smile\!\!\!\!\!\mid}^{+}_{A} B$.

Extension. Suppose $A \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} B$ and $D \supseteq B$ is given. We need to find $A' \equiv_{BC} A$ with $A' \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} D$. As usual we may assume $C \subseteq A, B$ and that these parameter sets are acl_+-closed. Let K be a small $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing all of these sets. Use extension for $\mathop{\smile\!\!\!\!\!\mid}^{0}$ to find $A' \mathop{\smile\!\!\!\!\!\mid}^{0}_{C} K$ with $\mathrm{tp}_0(A'/B) = \mathrm{tp}_0(A/B)$. This $\mathcal{L}_0$-isomorphism induces an $\mathcal{L}$-isomorphic structure on A'. By the derivation-like axiom, there is some $M \models T$ such that $M \leq_{\mathcal{L}_0} \mathcal{U}_0$ with $A', K \leq_{\mathcal{L}} M$. Since T_+ is the model companion of T, extend M to some $N \models T_+$, and let $\phi\colon N \to \mathcal{U}_+$ be an $\mathcal{L}$-elementary embedding of N in $\mathcal{U}_+$ which fixes K. Then

$$\mathrm{qftp}_+^{\mathcal{U}_+}(A/B) = \mathrm{qftp}_+^{K}(A/B) = \mathrm{qftp}_+^{M}(A'/B) = \mathrm{qftp}_+^{N}(A'/B) = \mathrm{qftp}_+^{\mathcal{U}_+}(\phi(A')/B).$$

By quantifier elimination for T_+, $\mathrm{tp}_+(A/B) = \mathrm{tp}_+(\phi(A')/B)$.

As usual,

$$\mathrm{qftp}_0^{\mathcal{U}_0}(A'/K) = \mathrm{qftp}_0^{M}(A'/K) = \mathrm{qftp}_0^{N}(A'/K) = \mathrm{qftp}_0^{\mathcal{U}_+}(\phi(A')/K) = \mathrm{qftp}_0^{\mathcal{U}_0}(\phi(A')/K),$$

and by quantifier elimination for T_0 and invariance and monotonicity for $\mathop{\smile\!\!\!\!\!\mid}^{0}$, we get $\phi(A') \mathop{\smile\!\!\!\!\!\mid}^{0}_{C} D$.

For the "In addition" clause in the statement, note that the arguments provided in (2), (3), (7) hold when working over models. □

Using the above theorem, we observe that rosiness transfers.

Corollary 2.12. *Assume both T_0 and T_+ admit elimination of imaginaries. If T_0 is rosy, then so is T_+.*

Proof. By Theorem 1.2(iv), it suffices to show that T_+ admits an strict independence relation. Taking $\mathop{\smile\!\!\!\!\!\mid}^{0}$ to be any strict independence relation (which exists by rosiness of T_0), Theorem 2.11 yields that $\mathop{\smile\!\!\!\!\!\mid}^{+}$ is an independence relation; thus, it suffices to show that $\mathop{\smile\!\!\!\!\!\mid}^{+}$ satisfies anti-reflexivity. Suppose $a \mathop{\smile\!\!\!\!\!\mid}^{+}_{C} a$. Then, by symmetry and monotonicity of $\mathop{\smile\!\!\!\!\!\mid}^{0}$, we get $a \mathop{\smile\!\!\!\!\!\mid}^{0}_{\mathrm{acl}_+(C)} a$; and so, by anti-reflexity of $\mathop{\smile\!\!\!\!\!\mid}^{0}$, we obtain

$$a \in \mathrm{acl}_0(\mathrm{acl}_+(C)) \cap \mathcal{U}_+ = \mathrm{acl}_+(C).$$

Where the last equality uses Lemma 2.7. □

We now address the transfer of the independence theorem.

Theorem 2.13. *Let $M \models T_+$ and suppose $T_0 \subseteq T$. Assume the following: T_0 has, in addition, an independence relation $\mathop{\smile\!\!\!\!\!\mid}^{1}$ such that $\mathop{\smile\!\!\!\!\!\mid}^{0}_{M} \implies \mathop{\smile\!\!\!\!\!\mid}^{1}_{M}$; T is derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\mid}^{1})$; and $\mathop{\smile\!\!\!\!\!\mid}^{0}_{M}$ satisfies monotonicity, symmetry, and extension. If $\mathop{\smile\!\!\!\!\!\mid}^{0}$ satisfies the independence theorem over M, then so does $\mathop{\smile\!\!\!\!\!\mid}^{+}$.*

Proof. Let $M \models T_+$, $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^{+}_{M} A_2$, $a_1 \mathop{\smile\!\!\!\!\!\!\mid}^{+}_{M} A_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^{+}_{M} A_2$, and $\mathrm{tp}_+(a_1/M) = \mathrm{tp}_+(a_2/M)$. We will show that there is $a \mathop{\smile\!\!\!\!\!\!\mid}^{+}_{M} A_1A_2$ realising $\mathrm{tp}_+(a_1/MA_1) \cup \mathrm{tp}_+(a_2/MA_2)$. Let $N \models T_+$ be some $\mathcal{L}$-elementary substructure of $\mathcal{U}_+$ containing all of the above subsets.

Note that, by Theorem 2.11, $\mathop{\smile\!\!\!\!\!\!\mid}^{+}_{M}$ satisfies symmetry and extension (the fact that T is derivation-like w.r.t. the independence relation $\mathop{\smile\!\!\!\!\!\!\mid}^{1}$ yields that T_+ has q.e. by Corollary 2.6). Thus, we may assume that A_1, A_2, a_1, and a_2 are all acl_+-closed and contain M (note that $T_0 \subseteq T$ implies that they are also acl_0-closed).

Claim 1. There is some $a \in \mathcal{U}_0$ with $a \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} N$ and $a \models \mathrm{tp}_0(a_1/A_1) \cup \mathrm{tp}_0(a_2/A_2)$.
Proof of claim. Note first that by definition of $\mathop{\smile\!\!\!\!\!\!\mid}^{+}$, we have the following facts: $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} A_2$, $a_1 \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} A_1$, $a_2 \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} A_2$, and $\mathrm{tp}_0(a_1/M) = \mathrm{tp}_0(a_2/M)$. By the independence theorem for $\mathop{\smile\!\!\!\!\!\!\mid}^{0}$, there is $a \in \mathcal{U}_0$ with $a \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} A_1A_2$ and $a \models \mathrm{tp}_0(a_1/A_1) \cup \mathrm{tp}_0(a_2/A_2)$. Now by extension for $\mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M}$, we may assume that $a \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} N$.

Claim 2. Inside $\mathcal{U}_0$, there are $\mathcal{L}_0$-isomorphic copies of N, N_1 and N_2, both containing a, with $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} N_2$ and $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1A_2} N_1N_2$.
Proof of claim. Note first that, by assumption and the fact that $a \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{M} N$, we have that $a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{M} N$. Now for $i = 1, 2$, let N_i' be the copy of N coming from the $\mathcal{L}_0$-automorphism $A_ia_i \mapsto A_ia$. By full existence for $\mathop{\smile\!\!\!\!\!\!\mid}^{1}$, let $N_i \equiv^0_{A_ia} N_i'$ with $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1a} N$ and $N_2 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2a} NN_1$. Then $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1} N_1$ and $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2} N_2$ by transitivity. From $a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{M} N$ we get $a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1} A_2$, and so $A_1a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1} A_2$. Along with $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{M} A_2$, transitivity gives $A_1a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{M} A_2$, so that $A_1a \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} A_2$ by base monotonicity. This implies $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} A_2$ and $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} A_2$. This last part implies $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} A_2a$ and along with $N_2 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2a} NN_1$ implies $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} N_2$. Also, $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1A_2} N_1$ by base monotonicity since $A_1A_2 \subseteq N$. From $NN_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2a} N_2$, we get $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2aN_1} N_2$, and hence $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_2N_1} N_2$ since $a \in N_1$. Combining this with $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1A_2} N_1$ gives $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1A_2} N_1N_2$.

Claim 3. There is some model of T which is an $\mathcal{L}$-extension of N, N_1, and N_2.
Proof of claim. Define $\mathcal{L}$-structures on N_1 and N_2 such that (N_i, A_i, a) is $\mathcal{L}$-isomorphic to (N, A_i, a_i). So $N_i \models T_+$ for $i = 1, 2$. Note that since a_i is an $\mathcal{L}$-substructure of N, a is also an $\mathcal{L}$-substructure of N_i. Now $N_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{a} N_2$, and a is relatively acl_0-closed in N_1 and N_2; since T is derivation-like w.r.t. $(T_0, \mathop{\smile\!\!\!\!\!\!\mid}^{1})$, there is some $P \models T$ such that $P \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $N_1, N_2 \leq_{\mathcal{L}} P$. Since $T_0 \subseteq T$ and using part (ii) of the definition of derivation-like, we have $\mathrm{acl}_0(N_1N_2)$ is an $\mathcal{L}$-substructure of P. By the uniqueness clause of part (ii) of derivation-like and the fact that $A_1 \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{M} A_2$, we have that $\mathrm{acl}_0(A_1A_2)$ is equipped with an $\mathcal{L}$-structure that makes it simultaneously an $\mathcal{L}$-substructure of N and an $\mathcal{L}$-substructure of $\mathrm{acl}_0(N_1N_2)$. Now $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{A_1A_2} N_1N_2$, and so $N \mathop{\smile\!\!\!\!\!\!\mid}^{1}_{\mathrm{acl}_0(A_1A_2)} \mathrm{acl}_0(N_1N_2)$ by invariance, base monotonicity, monotonicity, transitivity, and full existence. Now by part (i) of derivation-like we may find some $S \models T$ with $S \leq_{\mathcal{L}_0} \mathcal{U}_0$ and $N, \mathrm{acl}_0(N_1N_2) \leq_{\mathcal{L}} S$. So S is an $\mathcal{L}$-extension of N, N_1, and N_2, as desired.

Now S extends to some $S' \models T_+$. Let $j\colon S' \to \mathcal{U}_+$ be an $\mathcal{L}$-elementary embedding of S' in $\mathcal{U}_+$ that fixes N. Then

$$\mathrm{tp}_+^{\mathcal{U}_+}(a_1/A_1) = \mathrm{tp}_+^{N}(a_1/A_1) = \mathrm{tp}_+^{N_1}(a/A_1) = \mathrm{tp}_+^{S'}(a/A_1) = \mathrm{tp}_+^{\mathcal{U}_+}(j(a)/A_1)$$

and similarly we have $j(a) \equiv^+_{A_2} a_2$.

As T_0 has q.e., we have $\mathrm{tp}_0(a/N) = \mathrm{tp}_0(j(a)/N)$. Now, by construction of a, we had $a \mathop{\smile\!\!\!\!\!\!|}^0_M N$, and by monotonicity and invariance, we get $j(a) \mathop{\smile\!\!\!\!\!\!|}^0_M \mathrm{acl}(A_1A_2)$, and so $j(a) \mathop{\smile\!\!\!\!\!\!|}^+_M A_1A_2$. □

The following addresses transfer of stationarity.

Theorem 2.14. *Let $M \models T_+$ with $\mathrm{dcl}_0(M) \models T_0$. Suppose $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies base monotonicity, extension, and full existence. If $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies stationarity over M, then so does $\mathop{\smile\!\!\!\!\!\!|}^+$.*

Proof. Note that, by full existence and Corollary 2.6, T_+ has q.e.. Now suppose $M \subset N \subset \mathcal{U}_+$, $a, b \in \mathcal{U}_+$ with $\mathrm{tp}_+(a/M) = \mathrm{tp}_+(b/M)$, $a \mathop{\smile\!\!\!\!\!\!|}^+_M N$, and $b \mathop{\smile\!\!\!\!\!\!|}^+_M N$. Since $\mathop{\smile\!\!\!\!\!\!|}^+$ satisfies extension (by Theorem 2.11(7)), we may assume that N is a model of T_+. Let $K_a = \mathrm{acl}_+(Ma)$ and $K_b = \mathrm{acl}_+(Mb)$. By definition of $\mathop{\smile\!\!\!\!\!\!|}^+$, we have that $K_a \mathop{\smile\!\!\!\!\!\!|}^0_M N$ and $K_b \mathop{\smile\!\!\!\!\!\!|}^0_M N$. By extension for $\mathop{\smile\!\!\!\!\!\!|}^0$, the same independence holds after replacing N for some N_0 containing $N \cup \mathrm{dcl}_0(M)$. Hence, by base monotonicity, $K_a \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{dcl}_0(M)} N_0$ and $K_b \mathop{\smile\!\!\!\!\!\!|}^0_{\mathrm{dcl}_0(M)} N_0$. Note that $\mathrm{tp}_0(K_a/\mathrm{dcl}_0(M)) = \mathrm{tp}_0(K_b/\mathrm{dcl}_0(M))$. Then by stationarity for $\mathop{\smile\!\!\!\!\!\!|}^0$,

$$\mathrm{tp}_0(K_a/N) = \mathrm{tp}_0(K_b/N).$$

This implies that there is an $\mathcal{L}_0$-isomorphism $\langle K_aN\rangle_{\mathcal{L}_0} \to \langle K_bN\rangle_{\mathcal{L}_0}$ taking $a \mapsto b$ and fixing N.

Note that M is an $\mathcal{L}$-substructure of K_a, K_b, and N. Furthermore, by Remark 2.2 and full existence, $\mathrm{acl}_0(M) \cap \mathcal{U}_+ = \mathrm{acl}_+(M) = M$ (as $M \models T_+$).

By the derivation-like axiom, $\langle K_aN\rangle_{\mathcal{L}_0}$ and $\langle K_bN\rangle_{\mathcal{L}_0}$ are $\mathcal{L}$-substructures of $\mathcal{U}_+$. By its uniqueness clause, this $\mathcal{L}_0$-isomorphism must be an $\mathcal{L}$-isomorphism. So $\mathrm{qftp}_+(a/N) = \mathrm{qftp}_+(b/N)$. By quantifier elimination for T_+, we have $\mathrm{tp}_+(a/N) = \mathrm{tp}_+(b/N)$. □

Corollary 2.15. *Suppose $\mathop{\smile\!\!\!\!\!\!|}^0$ is nonforking independence.*

1. *Assume that $\mathrm{dcl}_0(M) \models T_0$ whenever $M \models T_+$. If T_0 is stable, then T_+ is stable and $\mathop{\smile\!\!\!\!\!\!|}^+$ is nonforking independence.*
2. *Assume $T_0 \subseteq T$. If T_0 is simple, then T_+ is simple and $\mathop{\smile\!\!\!\!\!\!|}^+$ is nonforking independence.*

Proof. (1) follows from Theorems 2.11 and 2.14; while (2) follows from Theorems 2.11 and 2.13 (note that in the latter we take $\mathop{\smile\!\!\!\!\!\!|}^1 = \mathop{\smile\!\!\!\!\!\!|}^0$). □

We now aim to prove a similar result on the transfer of NSOP_1. We will need to restrict to the case of fields to apply Theorem 2.13 with a particular choice of $\mathop{\smile\!\!\!\!\!\!|}^1$.

Assume T_0 is an $\mathcal{L}_0$-theory of fields, we say that a relation $\mathop{\smile\!\!\!\!\!\!|}^1$ on T_0 *implies $\mathcal{L}_0$-compositums* if for all $K, L \leq_{\mathcal{L}_0} \mathcal{U}_0$ satisfying $K \mathop{\smile\!\!\!\!\!\!|}^1_E L$, for some $E = \mathrm{acl}_0(E) \cap K =$

$\mathrm{acl}_0(E) \cap L$, the compositum $K \cdot L$ is an $\mathcal{L}_0$-substructure of $\mathcal{U}_0$. Following [9], we say that T_0 is *very $\mathcal{L}_0$-slim* if for every $F \leq_{\mathcal{L}_0} \mathcal{U}_0$ we have that

$$\mathrm{acl}_0(F) = F^{\mathrm{alg}} \cap \mathcal{U}_0.$$

Define the relation $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ on small subsets of $\mathcal{U}_0$ by

$$A \mathop{\smile\!\!\!\!\!\!\mid}\limits^{1}_{C} B \iff \langle A\, C\rangle_{\mathcal{L}_0} \mathop{\smile\!\!\!\!\!\!\mid}\limits^{\mathrm{alg}}_{\langle C\rangle_{\mathcal{L}_0}} \langle B\, C\rangle_{\mathcal{L}_0} \tag{1}$$

where $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{alg}}$ denotes algebraic independence in fields.

Fact 2.16. *Assume $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ implies $\mathcal{L}_0$-compositums. The relation $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ as defined above is an independence relation if and only if T_0 is very $\mathcal{L}_0$-slim.*

The proof is an adaptation of Theorem 2.1 of [9]. Some details are provided in Lemma 4.4.7 of the second author's thesis [13].

Corollary 2.17. *Assume that T_0 is very $\mathcal{L}_0$-slim, that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ implies $\mathcal{L}_0$-compositums, that T is derivation-like with respect to $(T_0, \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1)$, and that $T_0 \subseteq T$. If T_0 is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^0$ is Kim-independence, then T_+ is NSOP$_1$ and $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^+$ is Kim-independence.*

Proof. By Proposition 3.9.26 of [17], if two subfields are Kim-independent over a submodel, then they are algebraically independent. So the condition $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^0_M \implies \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1_M$ holds, and $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^+$ satisfies the independence theorem over models by Theorem 2.13 (noting that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ is an independence relation by Fact 2.16). Existence over models and chain local character each transfer from $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^0$ to $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^+$ since every model of T_+ is also a model of T_0. The remaining conditions of Theorem 1.2(iii) hold by Theorem 2.11 (note that (7) of that theorem, i.e. transfer of existence, does apply as T_+ has q.e., this is by Corollary 2.6 and the fact that $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^1$ satisfies full existence). □

Remark 2.18. One can readily check that when $T_0 \subseteq T_+$, all results of this section continue to hold after weakening condition (ii) of derivation-like by restricting to those D that are dcl_0-closed.

3. Examples

In this section we observe that there are plenty of examples of theories that are derivation-like, and hence to which the results of the previous section apply (when the model companion exists).

3.1. **Separably closed fields and Hasse-Schmidt fields.** Fix a prime $p > 0$ and e a nonnegative integer (finite). Let $\mathcal{L}_0$ be the language of fields, $T_0 = \mathrm{ACF}_p$ and $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^0$ forking-independence (which coincides with algebraic disjointness $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{alg}}$). Let $\mathcal{L}_{b,\lambda}$ be the language of fields expanded by constants $b = (b_1, \ldots, b_e)$ and unary function symbols $(\lambda_i)_{i \in p^e}$. Let $T_{b,\lambda}$ be the theory of fields of characteristic p together with sentences specifying that b is a p-basis and that the λ_i's are interpreted as the λ-functions with respect to b (in some fixed order of the p-monomials).

Lemma 3.1. *The theory $T_{b,\lambda}$ is derivation-like with respect to $(\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\mathrm{alg}})$.*

Proof. With $\mathcal{U}_0$ a monster model of ACF_p, let $A, B, C \models T_{b,\lambda}^{\forall}$ be as in the definition of derivation-like. Since $C \leq_{\mathcal{L}_{b,\lambda}} A$, we have that A/C is a separable field extension. This, together with the fact that $C = C^{\mathrm{alg}} \cap A$, implies that the field extension A/C is regular (i.e., separable and relatively algebraically closed). This, together with $A \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}_C B$, implies that A and B are linearly disjoint over C. Linear disjointness implies that b is p-independent in the compositum $A \cdot B$ (and hence a p-basis). If follows that $A \cdot B \models T_{b,\lambda}$ and $A, B \leq_{\mathcal{L}_{b,\lambda}} A \cdot B \leq_{\mathcal{L}_0} \mathcal{U}_0$. This shows condition (i) of derivation-like. Condition (ii) follows from the fact that p-bases are preserved when passing to separably algebraic extensions. □

The model companion of $T_{b,\lambda}$ is $\mathrm{SCF}_{p,e}$. Note that in this case for any $M \models \mathrm{SCF}_{p,e}$ we have that $\mathrm{dcl}^{\mathrm{ACF}_p}(M) \models \mathrm{ACF}_p$ (since perfect closures of separably closed fields remain separably closed). Thus, our Corollary 2.15(1) applies and recovers the well known fact that $\mathrm{SCF}_{p,e}$ is stable and (in the language $\mathcal{L}_{b,\lambda}$) forking-independece coincides with algebraic-disjointness.

In a similar fashion we can also recover the context of iterative Hasse-Schmidt derivations from [21]. Let $\mathcal{L}_\partial$ be the language of fields expanded by unary function symbols

$$((\partial_{1,j})_{j=1}^{\infty}, \dots, (\partial_{e,j})_{j=1}^{\infty}).$$

Let T_∂ be the theory of fields of characteristic p expanded by sentences specifying that $(\partial_{i,j})_{j=1}^{\infty}$ is an iterative Hasse-Schmidt derivation and that, for different i, they pairwise commute.

Lemma 3.2. *The theory T_∂ is derivation-like with respect to* $(\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}})$.

Proof. Let $A, B, C \models T_\partial^{\forall}$ be as the definition of derivation-like. By Lemma 2.3 and 2.4 from [21], after possibly passing to a purely inseparable extension of the separable closure of C, we may assume that C is strict and separably closed. Strictness implies that A/C is a separable extension. Thus, since C is separably closed, A/C is a regular field extension which implies that A and B are linearly disjoint over C. It follows that $A \cdot B$ is isomorphic to the quotient field of $A \otimes_C B$. The Hasse-Schmidt derivations extend uniquely to $A \otimes_C B$ and this yields an $\mathcal{L}_\partial$-structure on $A \cdot B$ making it a model of T_∂ (see for instance Lemma 2.5 of [21]). This yields condition (i) of derivation-like. Since Hasse-Schmmidt fields have a smallest strict extension (see [21, Lemma 2.4]) and separably algebraic extensions are étale, condition (ii) of derivation-like follows. □

The model companion of T_∂ is $\mathrm{SCH}_{p,e}$ (using the notation from [21]). Recall that the latter is the theory of fields equipped with strict and iterative Hasse-Schmidt derivations that pairwise commute and whose underlying field is a model of $\mathrm{SCF}_{p,e}$. As in the SCF case above, for any $M \models \mathrm{SCH}_{p,e}$ we have that $\mathrm{dcl}^{\mathrm{ACF}_p}(M) \models \mathrm{ACF}_p$. Thus, Corollary 2.15(1) applies and recovers the fact that $\mathrm{SCH}_{p,e}$ is stable and in the language $\mathcal{L}_\partial$ forking-independence coincides with algebraic-disjointness.

3.2. **$\mathcal{D}$-fields in characteristic zero.** Let $\mathcal{L}_0$ be an expansion of the field language and T_0 a complete and model-complete $\mathcal{L}_0$-theory of fields of characteristic zero. As before, we denote by $\mathop{\smile\!\!\!\!\!\!\mid}^{0}$ an invariant ternary relation on a monster model $\mathcal{U}_0 \models T_0$. Recall that $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{alg}}$ denotes the algebraic disjointness relation.

We say that $\mathop{\smile\!\!\!\!\!\!|}^0$ *implies algebraic-disjointness* if

$$K \underset{E}{\overset{0}{\downfree}} L \implies K \underset{E}{\overset{\mathrm{alg}}{\downfree}} L$$

for K, L $\mathcal{L}_0$-substructures of $\mathcal{U}_0$ and E a common $\mathcal{L}_0$-substructure of K and L. Recall from the previous section that $\downfree^0$ *implies $\mathcal{L}_0$-compositums* if for all $K, L \leq_{\mathcal{L}_0} \mathcal{U}_0$ satisfying $K \downfree^0_E L$, for some $E = \mathrm{acl}_0(E) \cap K = \mathrm{acl}_0(E) \cap L$, the compositum $K \cdot L$ is an $\mathcal{L}_0$-substructure of $\mathcal{U}_0$. Recall also that T_0 is *very $\mathcal{L}_0$-slim* if for every $F \leq_{\mathcal{L}_0} \mathcal{U}_0$ we have that

$$\mathrm{acl}_0(F) = F^{\mathrm{alg}} \cap \mathcal{U}_0.$$

Following the general framework of Moosa-Scanlon from [16], let $\mathcal{D}$ be a finite-dimensional algebra over a field k of characteristic zero equipped with a k-basis $\epsilon_0 = 1, \epsilon_1, \ldots, \epsilon_d$ such that $\mathcal{D}$ is a local ring with residue field k. A $\mathcal{D}$-field K is a field which is also a k-algebra equipped with a sequence of operators $(\partial_i : K \to K)_{i=1}^d$ such that the map $K \to K \otimes_k \mathcal{D}$ defined by

$$a \mapsto a \otimes \epsilon_0 + \partial_1(a) \otimes \epsilon_1 + \cdots \partial_d(a) \otimes \epsilon_d$$

is a k-algebra homomorphism. Let $\mathcal{L}_{\mathcal{D}}$ be the language $\mathcal{L}_0$ expanded by the language of k-algebras and the unary function symbols $\{\partial_1, \ldots, \partial_d\}$. Let $T_{\mathcal{D}}$ be $\mathcal{L}_{\mathcal{D}}$-theory consisting of T_0 together with the theory of $\mathcal{D}$-fields. In addition, let $T_{\mathcal{D}^*}$ be $T_{\mathcal{D}}$ expanded by sentences specifying that the ∂_i's pairwise commute.

Remark 3.3. Let $\mathcal{D} = \mathbb{Q}[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^2$. In this case, the theory $T_{\mathcal{D}}$ is the theory of differential fields of characteristic zero with d-many (not necessarily commuting) derivations whose underlying field is a model of T_0. The theory $T_{\mathcal{D}^*}$ is similar but requires the derivations to pairwise commute.

Lemma 3.4. *Suppose $\downfree^0$ implies algebraic-disjointness and $\mathcal{L}_0$-compositums. Also assume T_0 is very $\mathcal{L}_0$-slim. Then, the theories $T_{\mathcal{D}}$ and $T_{\mathcal{D}^*}$ are derivation-like with respect to $(T_0, \downfree^0)$.*

Proof. First we prove $T_{\mathcal{D}}$ is derivation-like. Let $K, L, E \models T_{\mathcal{D}}^{\forall}$ be as in the definition of derivation-like. Since $E = \mathrm{acl}_0(E) \cap K$ and T_0 is very $\mathcal{L}_0$-slim, K/E is a regular field extension. Since $\downfree^0$ implies algebraic disjointness, it follows that K and L are linearly disjoint over E. Then $K \cdot L$ is isomorphic to the quotient field of $K \otimes_E L$. Since $\downfree^0$ implies $\mathcal{L}_0$-compositums and $\mathcal{D}$-structures extend uniquely to $K \otimes_E L$ (see [2, Proposition 2.20]), this yields an $\mathcal{L}_{\mathcal{D}}$-structure on $K \cdot L$. As we are in characteristic zero, this $\mathcal{D}$-structure extends to all of $\mathcal{U}_0$ (recall that $\mathcal{D}$-structures always extend to smooth extensions, see [2, Lemma 2.7(3)]) which yields condition (i) of derivation-like. Since algebraic extensions are étale (in characteristic zero), condition (ii) follows (recall that $\mathcal{D}$-structures extend uniquely to étale extensions, see [2, Lemma 2.7(2)]).

For $T_{\mathcal{D}^*}$, the same argument works by simply noting that uniqueness of the $\mathcal{D}$-structure on $K \otimes_E L$ forces the ∂_i's to commute. And similarly when passing to algebraic extensions (as they are étale in characteristic zero). To extend the $\mathcal{D}$-structure from $K \cdot L$ to $\mathcal{U}_0$, first extend to a transcendence basis in a trivial way to force commutativity of the ∂_i's and after this the unique extension to $\mathcal{U}_0$ will necessarily commute. This sort of argument is spelled out in Example 4.4.11 of the second author's thesis [13]. $\square$

For the remainder of this section we set $\mathop{\smile\!\!\!\!\!|}^0$ to be the relation we defined in (1) at the end of Section 2.

$$A \mathop{\downarrow}^{0}_{C} B \iff \langle A\,C\rangle_{\mathcal{L}_0} \mathop{\downarrow}^{\mathrm{alg}}_{\langle C\rangle_{\mathcal{L}_0}} \langle B\,C\rangle_{\mathcal{L}_0} \tag{2}$$

Note that this particular relation implies algebraic-disjointness. We recall Fact 2.16 along with an additional fact.

Fact 3.5. (1) *Assume $\downarrow^0$ implies $\mathcal{L}_0$-compositums. The relation $\downarrow^0$ as defined in (2) is an independence relation if and only if T_0 is very $\mathcal{L}_0$-slim.*
(2) *Suppose $\mathcal{L}_0 = \mathcal{L}_{fields}(C)$ where C is a set of constants. If models of T_0 are large fields, then T_0 is very $\mathcal{L}_0$-slim.*

We note that (2) is an adaptation of Theorem 5.4 of [9] and appears in Lemma 4.4.10 of the second author's thesis [13].

Corollary 3.6. *Suppose models of T_0 are large fields, $\mathcal{L}_0 = \mathcal{L}_{fields}(C)$ for C a set of constants, and $\downarrow^0$ is given as in (2). Assume $T_{\mathcal{D}}$ and $T_{\mathcal{D}^*}$ have model companions $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$, respectively.*

(i) *If T_0 is simple, then so are $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$.*
(ii) *If T_0 is NSOP$_1$, then so are $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$.*
(iii) *Assume T_0, $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$ all eliminate imaginaries. If T_0 is rosy, then so are $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$.*

Proof. (i) follows immediately by Corollary 2.15(2), (ii) by Corollary 2.17, (iii) by Corollary 2.12. □

Remark 3.7. Under the hypothesis of Corollary 3.6 (and recall that we are in characteristic zero), an immediate application is that

(i) if T_0 is the theory of a bounded PAC field , then $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$ are simple,
(ii) if T_0 is the theory of an ω-free PAC field, then $T^*_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$ are NSOP$_1$, and
(iii) the theory CODF (closed ordered differential field in one derivation) is rosy.

Indeed, for (i) recall that bounded PAC fields are simple; while for (ii) recall that ω-free perfect PAC fields are NSOP$_1$. For (iii), recall that the theory RCF is rosy and eliminates imaginaries; also, CODF eliminates imaginaries by [6].

We conclude this section by noting that under the hypothesis of Corollary 3.6 the model companions $T^+_{\mathcal{D}}$ and $T^+_{\mathcal{D}^*}$ in fact exist. Existence of $T^+_{\mathcal{D}}$ is one of the main results of the second author's paper [12] and, moreover, the simplicity claimed in Remark 3.7(i) already appears there. Existence of $T^+_{\mathcal{D}^*}$ will appear in a forthcoming paper.[1] In the case when $\mathcal{D} = \mathbb{Q}[x_1, \dots, x_d]/(x_1, \dots x_d)^2$ (i.e., in the context of differential fields of characteristic zero, see Remark 3.3) and T_0 is the theory of a bounded PAC field, the existence of $T^+_{\mathcal{D}^*}$ is an instance of the main result of [19] and its simplicity already appears in [7].

[1] As part of joint work of the first author with Jan Dobrowolski.

3.3. **Differential fields in positive characteristic.** In this section we apply our results in the context of separably differentially closed fields [8]. Fix a prime $p>0$. Let $\mathcal{L}_\lambda$ be the language of fields expanded by the (countably-many) λ-functions. Namely, by functions $(\lambda_{n,i} : n \in \omega, i \in p^n)$ where $\lambda_{n,i}$ is $(n+1)$-ary. We denote by $\mathrm{SCF}^\lambda_{p,\infty}$ the theory of separably closed fields of infinite (algebraic) degree of imperfection expanded by sentences specifying that the $\lambda_{n,i}$ are to be interpreted as the λ-functions; that is, for $(\bar a, b)$, if $\bar a$ is p-dependent or $(\bar a, b)$ is p-independent then $\lambda_{n,i}(\bar a, b) = 0$; otherwise,

$$b = \sum_{i \in p^n} (\lambda_{n,i}(\bar a, b))^p \, m_i(\bar a)$$

where the $m_i(\bar a)$'s denote the p-monomials (with some fixed order). We denote forking-independence in $\mathrm{SCF}^\lambda_{p,\infty}$ by $\mathop{\smile\!\!\!\!\!\mid}\,^0$, and $\mathcal{U}_0$ is a monster model. Recall from [18] that for $\mathcal{L}_\lambda$-substructures K, L, E of $\mathcal{U}_0$ we have

$$K \mathop{\smile\!\!\!\!\!\mid}^{0}_{E} L \qquad \Longleftrightarrow \qquad K \text{ and } L \text{ are algebraically disjoint and } p\text{-disjoint over } E.$$

Let $\mathcal{L}_{\lambda,\delta}$ be the expansion of $\mathcal{L}_\lambda$ by a (single) unary function symbol δ and let DF^λ_p be the theory of differential fields of characteristic p with sentences specifying that the $\lambda_{n,i}$ are the λ-functions.

Lemma 3.8. *The theory* DF^λ_p *is derivation like with respect to* $(\mathrm{SCF}^\lambda_{p,\infty}, \mathop{\smile\!\!\!\!\!\mid}\,^0)$.

Proof. Let $K, L, , E \models (\mathrm{DF}^\lambda_p)^\forall$ be as in the definition of derivation-like. Since $K \mathop{\smile\!\!\!\!\!\mid}^0_E L$, K and L are p-disjoint over E, and so $K \cdot L$ is an $\mathcal{L}_\lambda$-substructure of $\mathcal{U}_0$.

On the other hand, since $E \leq_{\mathcal{L}_{\lambda,\delta}} K$, we have that K/E is a separable field extension. This, together with the fact that $E = E^{\mathrm{alg}} \cap K$, implies that K/E is a regular field extension. This, together with $K \mathop{\smile\!\!\!\!\!\mid}^0_E L$, implies that K and L are linearly disjoint over E. Linear disjointness implies $K \cdot L$ is isomorphic to the quotient field of $K \otimes_E L$. This yields a derivation on $K \cdot L$ making it a model of DF^λ_p. This yields condition (i) of derivation-like. Since separably algebraic extensions are étale, condition (ii) follows. □

For $\epsilon \in \mathbb{N} \cup \{\infty\}$, recall that a differential field (K, δ) of characteristic p is said to have differential degree of imperfection ϵ if

$$[C_K : K^p] = p^\epsilon.$$

Here C_K denotes the field of δ-constants of (K, δ). When $\epsilon = \infty$ the above equality should be understood as the degree $[C_K : K^p]$ being infinite. See [8] for further details.

In [8] it was shown that DF^λ_p has a model-companion; namely, the theory $\mathrm{SDCF}^\lambda_{p,\infty}$ of separably differentially closed fields of characteristic p of infinite differential degree of imperfection expanded by the λ-functions. We note that in [8] the authors work in the language of the so-called differential λ-functions, denoted $\ell_{n,i}$, but the model-companiability result holds as well working with the algebraic λ-functions (the argument is spelled out in [13, Fact 4.4.16]). We note that $\mathrm{SCF}^\lambda_{p,\infty} \subseteq \mathrm{SDCF}^\lambda_{p,\infty}$. Thus, Corollary 2.15(1) applies and recovers the fact that $\mathrm{SDCF}^{\lambda}_{p,\infty}$ is stable; furthermore, it shows that, in the language $\mathcal{L}_{\lambda,\delta}$, forking-independence coincides with algebraic-disjointness and p-disjointness (which is not explicitly stated in [8]).

We conclude this section by noting that, unfortunately, our results do not seem to apply to the theory DCF_p, differentially closed fields of characteristic $p > 0$, studied by Wood in [20]. Recall that DCF_p is the model-companion of DF_p, the theory of differential fields (of characteristic p) in the language of differential fields $\mathcal{L}_\delta$. In this language the theory DCF_p does not eliminate quantifiers, but in [20] Wood showed that it suffices to add the *p-th root on constants* function ℓ_0; namely, the unique function satisfying

$$\begin{cases} (\ell_0(x))^p = x, & \text{when } \delta(x) = 0 \\ \ell_0(x) = 0, & \text{o.w.} \end{cases} \tag{3}$$

The theory of differentially perfect fields (i.e., those (K,δ) such that $C_K = K^p$) is denoted by $\mathrm{DPF}^{\ell_0}_p$ and can be axiomatised by expanding DF_p by a sentence specifying (3) above. It follows that $\mathrm{DCF}^{\ell_0}_p$ is the model-completion of $\mathrm{DPF}^{\ell_0}_p$. One could ask whether $\mathrm{DPF}^{\ell_0}_p$ is derivation-like with respect to $\mathrm{SCF}^{\lambda}_{p,\infty}$ (note that the underlying field of a DCF_p is a model of $\mathrm{SCF}_{p,\infty}$). We now prove this is not the case.

Lemma 3.9. *The theory* $\mathrm{DPF}^{\ell_0}_p$ *is* **not** *derivation-like with respect to* ($\mathrm{SCF}^{\lambda}_{p,\infty}, \mathop{\smile\!\!\!\!\!\!|}^{0}$)*.*

Proof. Consider the function field $K = \mathbb{F}_p(t)$ with standard derivation $\delta = \frac{d}{dt}$. Note that $(K,\delta) \models \mathrm{DPF}_p$ since $C_K = \mathbb{F}_p$. Inside the model $\mathcal{U}_0$ of $\mathrm{SCF}^{\lambda}_{p,\infty}$, find s such that $t \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathbb{F}_p} s$ and $\mathrm{tp}^{\mathrm{SCF}^{\lambda}_{p,\infty}}(t/\mathbb{F}_p) = \mathrm{tp}^{\mathrm{SCF}^{\lambda}_{p,\infty}}(s/\mathbb{F}_p)$. Equip $L = \mathbb{F}_p(s)$ with the derivation $\delta = \frac{d}{ds}$. We argue that there cannot be an M as in condition (i) of derivation-like. It there were, M would be a model of $\mathrm{DPF}^{\ell_0}_p$. In other words, $C_M = M^p$. Since $K \mathop{\smile\!\!\!\!\!\!|}^{0}_{\mathbb{F}_p} L$, we obtain that K and L are p-disjoint over $\mathbb{F}_p$; and so $K \cdot L = \mathbb{F}_p(t,s)$ is an $\mathcal{L}_\lambda$-substructure of M. Hence, the extension $M/\mathbb{F}_p(t,s)$ is separable, which implies that $\mathbb{F}_p(t,s)$ is differentially perfect. But, since $\delta(t-s) = 0$, this would imply that $t-s$ has a p-th root in $\mathbb{F}_p(t,s)$, which is impossible (as t and s are algebraically independent). $\square$

One could further ask whether $\mathrm{DPF}^{\ell_0}_p$ is derivation-like with respect to ACF_p. Again, this is not the case.

Lemma 3.10. *The theory* $\mathrm{DPF}^{\ell_0}_p$ *is* **not** *derivation-like with respect to* ($\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$)*.*

Proof. Consider the function field $K = \mathbb{F}_p(t)$ equipped with the standard derivation $\delta = \frac{d}{dt}$. Let x be a differential indeterminate over K. Let $s := t + x^p$. Then, the derivation on $M := K\langle x\rangle = K(x, \delta x, \dots)$ restricts to the standard derivation $\delta = \frac{d}{ds}$ on $L := \mathbb{F}_p(s)$. Note that both K and L are differentially perfect and $K \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}_{\mathbb{F}_p} L$. However, the algebraic closure of the compositum $K \cdot L$ contains x but it does not contain $\delta(x)$; namely, it is not a differential subfield. In other words, condition (ii) of derivation-like does not hold. $\square$

Remark 3.11.

(i) While DCF_p is a stable theory, the two proofs above show that forking-independence does not have an obvious algebraic description [2]. Indeed, the proof of Lemma 3.9 shows that $\mathop{\smile\!\!\!\!\!\!|}^{0} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{SCF}_{p,\infty}}$ does not satisfy full existence in DCF_p; while the proof of 3.10 shows that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}}$ does not satisfy base-monotonicity in DCF_p. Currently the authors are not aware of an algebraic description of forking-independence in this theory.

(ii) We leave it as an exercise to check that $\mathrm{DPF}_p^{\ell_0}$ is almost derivation-like with respect to $(\mathrm{ACF}_p, \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{alg}})$ in the sense of Remark 2.9, and hence Lemmas 2.5 and 2.7 apply to the theory $\mathrm{DCF}_p^{\ell_0}$.

3.4. **CCMs with meromorphic vector fields.** Our final example demonstrates that our results apply beyond theories of fields. Namely, we observe that the theory, recently formulated by Moosa [14], of compact complex manifolds equipped with a "differential" structure fits into our setup.

Recall that the theory CCM - compact complex manifolds - is the theory of the multi-sorted structure consisting of all compact complex manifolds (or rather all reduced and irreducible compact complex-analytic spaces) by naming as basic relations all closed complex-analytic subsets of finite cartesian products of sorts. See [15] for further details on this theory. However, in [14], Moosa works in the seemingly more general setup of "compactifiable" (rather than compact) complex-analytic spaces. Namely, he works in a definable-expansion of CCM where there is a sort for each irreducible meromorphic variety. We denote by $\mathcal{L}_0$ the language of this expansion and continue to denote the theory of the expanded structure by CCM. The advantage of this expansion is that now sorts are closed under taking tangent bundles. The reader might want to refer to [14, §2] for further details and explanations.

In the language $\mathcal{L}_\nabla = \mathcal{L}_0 \cup \{\nabla_S : S \text{ is a sort of } \mathcal{L}_0\}$, where each ∇_S is a function symbol from sort S to TS, Moosa considers the universal $\mathcal{L}_\nabla$-theory $\mathrm{CCM}_\nabla^\forall$ obtained by adding to $\mathrm{CCM}^\forall$ axioms specifying that $\nabla_S : S \to TS$ is a section to $\pi : TS \to S$ together with a compatibility condition of ∇ with definable meromorphic maps between sorts (see [14, Definition 3.3]). It turns out that, somewhat unintentionally, Moosa has proven that $\mathrm{CCM}_\nabla^\forall$ is derivation-like. Namely,

Lemma 3.12. *The theory* $\mathrm{CCM}_\nabla^\forall$ *is derivation-like with respect to* $(\mathrm{CCM}, \mathop{\smile\!\!\!\!\!\!|}^{0})$ *(here* $\mathop{\smile\!\!\!\!\!\!|}^{0}$ *denotes forking independence).*

Proof. In [14, Lemma 6.2] Moosa proved a form of independent amalgamation that readily yields condition (i) of derivation-like. In addition, in [14, Lemma 6.1], he proves the uniqueness of differential CCM-structures of dcl-closed sets inside $\mathrm{acl}^{\mathrm{CCM}}$-closures, yielding condition (ii) of derivation-like (or rather the weakening observed in Remark 2.18). □

[2] These examples grew out of discussions with Amador Martin-Pizarro.

In [14, Theorem 5.5], Moosa proves that the theory $\mathrm{CCM}_{\nabla}^{\forall}$ admits a model-companion which he denotes by DCCM. Our results then yield some of the model-theoretic properties of DCCM deduced in §6 and §7 of [14]; e.g., completeness, quantifier elimination, description acl and dcl, and stability.

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