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Waves in a two-component model of galactic dynamo: Metastability and stochastic generation

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Abstract

This paper is concerned with the stochastic generation of a large-scale galactic magnetic field and propagation of magnetic fronts in the subcritical regime. Starting with a two-component thin-disk $\alpha\Omega$-dynamo model in the axisymmetric case, we derive the equation for order parameter in the form of a stochastic reaction–diffusion equation with free energy functional. This equation describes subcritical generation of galactic magnetic field as a first-order phase transition in spatially extended system. We consider in particular a situation in which the magnetic field propagates as a plane front. We derive an approximate formula for propagation speed.

Keywords: Waves; Galactic dynamo; Metastability

1. Introduction

The generation and propagation of magnetic fields in galaxies have been studied for many years [1–4]. Nonetheless, many fundamental aspects of the galactic dynamo remain to be understood. The propagation of magnetic fronts is one such aspect of dynamo evolution. It is now accepted that the generation of the large-scale magnetic field occurs as a result of the simultaneous action of differential rotation of the galactic disk and turbulent motions of the interstellar medium. Standard mathematical procedure consists of looking for exponentially growing solutions of the linear mean field dynamo equation (kinematic dynamo). The propagation of a magnetic front then can be analyzed in terms of the classical Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation for the azimuthal magnetic field [5,6]. This equation is a generic model describing front propagation into an unstable state [7]. If the dynamo excitation occurs within a certain radius $r\leqslant r_0$, then the magnetic front propagates into the unstable region $r>r_0$, where the linear growth rate γ is positive (*supercritical* case). One can find that the minimal propagation speed is $c=2(\gamma\beta)^{1/2}$, where β is a magnetic diffusivity. This type of magnetic front is referred to as an exterior front [6]. In fact, there are infinitely many possible wave velocities which are determined by initial conditions. The front-like initial

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condition for a magnetic field ensures the minimal rate of propagation c. Recently, these results have been extended to the case when the memory effects are taken into account. It has been shown that the integral turbulent transport leads to the essential decrease of the speed of magnetic front [8,9].

The essential feature of early models for magnetic waves is that the minimal propagation rate is found from linear analysis. The main purpose of this work is to consider the *subcritical* case when the propagation of a magnetic wave is essentially a nonlinear phenomenon. We are going to use recent results concerning nonnormal growth and nonlinear instability for the galactic dynamo [10,11]. The key insight gained from this theory is that although the trivial state with zero magnetic field is linearly stable, the non-normality due to differential rotation and the dependence of α-effect and turbulent magnetic diffusivity on magnetic field can lead to the instability with respect to *finite* perturbations. Thus, in the *subcritical* case the generation of large-scale magnetic field can be regarded as a stochastic nucleation in a spatially extended dissipative system. Depending upon relative stability of stationary states, the finite localized perturbations (nuclei) either grow or shrink. When they grow, the propagation of magnetic fronts is observed. It should be noted that although the phenomenon of metastability occurs in many different situations in physics, it has received little attention in the context of magnetic field generation. Let us mention the work of Tobias on hysteresis in solar dynamo [12] and the so-called self-killing and self-creating dynamos by Fuchs, Rädler, and Rheinhard [13].

In this paper we use the comparatively simple thin-disk asymptotic approach to axisymmetric mean field dynamo for disk galaxy [3]. We restrict ourselves to the *nonlinear* $\alpha\Omega$-dynamo when both alpha and magnetic diffusivity quenching are taken into account. The aim is to derive a closed stochastic partial differential equation for the order parameter describing the *subcritical* generation and propagation of magnetic fronts. We use a transformation of variables and a technique of adiabatic elimination [15,16]. Our intention is to describe the magnetic field generation as a stochastic process in a spatially extended system with multiple stationary states (a first-order phase transition) [17]. The magnetic front can be regarded as a trigger wave connecting an initial metastable state and absolutely stable state [7,18].

2. Thin-disk dynamo equation

We start with an axisymmetric turbulent dynamo in the galactic disk of thickness $2h$ and radius R that rotates with angular velocity $\Omega(r)$ ($R \gg h$) [1–3]. A mean field model for the evolution of the components of magnetic field $B_r(t,r)$ and $B_\varphi(t,r)$ can be written as

$$\frac{\partial B_r}{\partial t} = -\frac{\alpha(B_\varphi)B_\varphi}{h} - \frac{\pi^2\beta(B_\varphi)B_r}{4h^2} + \nabla(\beta(B_\varphi)\nabla B_r) + F_r(t,r),$$

$$\frac{\partial B_\varphi}{\partial t} = g_\Omega B_r - \frac{\pi^2\beta(B_\varphi)B_\varphi}{4h^2} + \nabla(\beta(B_\varphi)\nabla B_\varphi) + F_\varphi(t,r), \tag{1}$$

where $\alpha(B_\varphi)$ is the nonlinear function describing the α-effect, $\beta(B_\varphi)$ is the nonlinear magnetic diffusivity, $g_\Omega = r\,\mathrm{d}\Omega/\mathrm{d}r$ is the measure of differential rotation and ($\mathrm{d}\Omega/\mathrm{d}r < 0$), ∇ is the gradient operator in the polar system of coordinates: $\nabla B = \partial B/\partial r\,\mathbf{e}_r$. To take into account unresolved turbulent fluctuations, we add two stochastic terms $F_r(t,r)$ and $F_\varphi(t,r)$ on the right-hand side of (1) [19]. We average the magnetic field over the vertical cross-section of a turbulent disk and consider the spatial structure in the galactic plane only [3]. Such a 'no-z' model proves to be generic and provides a very reasonable match to observations of the magnetic field generation in disk-like galaxies [2]. Since $B_z/B_{r,\varphi} \sim h/R \ll 1$, we are only interested in the radial, B_r, and azimuthal, B_φ, components of the magnetic field $\mathbf{B}$. Here we introduce two functions $\alpha(B_\varphi)$ and $\beta(B_\varphi)$ describing the quenching mechanism. The current theories disagree about how the α-effect and turbulent diffusivity β are suppressed by the magnetic field. In this paper we use the following nonlinear functions:

$$\alpha(B_\varphi) = \alpha_0(1 + k_\alpha(B_\varphi/B_{eq})^2)^{-1}, \quad \beta(B_\varphi) = \beta_0\left(1 + \frac{k_\beta}{1 + (B_{eq}/B_\varphi)^2}\right)^{-1}, \tag{2}$$

where k_α and k_β are positive constants, B_{eq} is the equipartition strength (see, for example, Ref. [4], p. 799). It should be noted that whilst in this paper we choose specific forms of the dependence of α and β on the magnetic field B_φ, the core result is not dependent upon the precise forms of these functions. B_{eq} is defined as a field for which the magnetic energy is equal to the characteristic energy of the turbulent fluctuations: $B_{eq} = \rho v_T^2$. Here ρ is the density and v_T^2 is the characteristic velocity associated with the large-scale turbulent flow. Both functions $\alpha(B_\varphi)$ and $\beta(B_\varphi)$ decay with B_φ thus describing the negative feedback on the magnetic field generation; α_0 and β_0 are chosen in a such way that $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$. Note that the dependence of the magnetic diffusivity $\beta(B_\varphi)$ on the azimuthal component B_φ is crucial for the *subcritical* generation of magnetic field. More general forms of these functions were considered in Ref. [14].

The mean field dynamo equations (1) can be non-dimensionalized by using a characteristic length $2h/\pi$, an angular velocity Ω_0, and the equipartition strength B_{eq}:

$$\frac{\partial B_r}{\partial t} = -\delta\varphi_\alpha(B_\varphi)B_\varphi - \varepsilon\varphi_\beta(B_\varphi)B_r + \varepsilon\nabla(\varphi_\beta(B_\varphi)\nabla B_r) + f_r(t,r),$$

$$\frac{\partial B_\varphi}{\partial t} = -gB_r - \varepsilon\varphi_\beta(B_\varphi)B_\varphi + \varepsilon\nabla(\varphi_\beta(B_\varphi)\nabla B_\varphi) + f_\varphi(t,r), \tag{3}$$

where

$$\delta = \frac{R_\alpha}{R_\omega}, \quad \varepsilon = \frac{\pi^2}{4R_\omega}, \quad g = \frac{|g_\Omega|}{\Omega_0} \tag{4}$$

and R_α and R_ω are the dimensionless measures of relative strength of the α-effect and the differential rotation, respectively:

$$R_\alpha = \frac{\alpha_0 h}{\beta_0}, \quad R_\omega = \frac{\Omega_0 h^2}{\beta_0}. \tag{5}$$

The random forces $f_r(t,r) = F_r(t,r)/\Omega_0$ and $f_\varphi(t,r) = F_\varphi(t,r)/\Omega_0$ are assumed to be Gaussian delta-correlated random fields with zero-mean

$$\langle f_r(t,r)f_r(t',r')\rangle = D_r\delta(t-t')\delta(r-r'), \tag{6}$$

$$\langle f_\varphi(t,r)f_\varphi(t',r')\rangle = D_\varphi\delta(t-t')\delta(r-r'). \tag{7}$$

The nonlinear functions $\varphi_\alpha(B_\varphi)$ and $\varphi_\beta(B_\varphi)$ are:

$$\varphi_\alpha(B_\varphi) = \frac{1}{1+k_\alpha B_\varphi^2}, \quad \varphi_\beta(B_\varphi) = \frac{(1+B_\varphi^2)}{1+(k_\beta+1)B_\varphi^2}. \tag{8}$$

In this paper we consider only the case of $\alpha\Omega$-dynamo for which the differential rotation dominates over the α-effect: $\Omega_0 h \gg \alpha_0$, that is $R_\alpha \ll R_\omega$. It means that the system (3) involves two small parameters δ and ε. The typical values are $\delta = 0.01$ and $\varepsilon = 0.1$. For small values δ, ε and $g\sim 1$, the linearized operator in (3) is highly non-normal which might lead to a large transient growth of the azimuthal component B_φ. Comprehensive survey and many examples of non-normal systems are given in Ref. [21]. One can also expect a high sensitivity of the second moments of a magnetic field to the stochastic perturbations $f_r(t,r)$ and $f_\varphi(t,r)$ [10].

Linearization of a zero-dimensional dynamical system (3) about the equilibrium point $(0,0)$ shows that in the *subcritical* case when both eigenvalues λ_1 and λ_2 are negative:

$$\lambda_1 = -\varepsilon + \sqrt{g\delta}, \quad \lambda_2 = -\varepsilon - \sqrt{g\delta}, \tag{9}$$

the point $(0, 0)$ is a stable node. The corresponding eigenvectors are

$$\mathbf{h}_1 = (-\mu, 1)^{\mathrm{T}}, \quad \mathbf{h}_2 = (\mu, 1)^{\mathrm{T}}, \quad \mu = \sqrt{\frac{\delta}{g}} \ll 1. \tag{10}$$

3. Stochastic normal form of dynamo equation

In this paper we consider only the *subcritical* case ($\varepsilon > \sqrt{g\delta}$) [10]. It is convenient to represent the system (3) in the stochastic normal form [20]. By using the eigenvectors $\mathbf{h}_1$ and $\mathbf{h}_2$ as a basis, one can introduce the change of variables $(B_r, B_\varphi) \to (u, v)$:

$$B_r(t,r) = \mu(v(t,r) - u(t,r)), \quad B_\varphi(t,r) = v(t,r) + u(t,r). \tag{11}$$

The partial derivatives of the fields $u(t,r)$ and $v(t,r)$ are

$$\frac{\partial u}{\partial t} = \frac{1}{2\mu}\left(-\frac{\partial B_r}{\partial t} + \mu\frac{\partial B_\varphi}{\partial t}\right),$$

$$\frac{\partial v}{\partial t} = \frac{1}{2\mu}\left(\frac{\partial B_r}{\partial t} + \mu\frac{\partial B_\varphi}{\partial t}\right), \tag{12}$$

and the nonlinear stochastic system (3) can be rewritten as:

$$\frac{\partial u}{\partial t} = \frac{1}{2\mu}[\delta(\varphi_\alpha - 1)v + (\delta - 2\mu\varepsilon\varphi_\beta + \delta\varphi_\alpha)u] + \varepsilon\nabla(\varphi_\beta\nabla u) - \frac{1}{2\mu}f_r + \frac{1}{2}f_\varphi,$$

$$\frac{\partial v}{\partial t} = \frac{1}{2\mu}[(-2\mu\varepsilon\varphi_\beta - \delta\varphi_\alpha - \delta)v + \delta(1 - \varphi_\alpha)u] + \varepsilon\nabla(\varphi_\beta\nabla v) + \frac{1}{2\mu}f_r + \frac{1}{2}f_\varphi, \tag{13}$$

where

$$\varphi_\alpha = \varphi_\alpha(v + u), \quad \varphi_\beta = \varphi_\beta(v + u). \tag{14}$$

Since the parameter μ is small, the stochastic term $f_\varphi(t,r)$ can be neglected compared to $(2\mu)^{-1}f_r(t,r)$. Since the latter term is proportional to the large parameter μ^{-1}, it explains the sensitivity of the non-normal dynamical systems to random perturbations (see Ref. [21]). In the linear case, the system (13) can be rewritten in a decoupled form:

$$\frac{\partial u}{\partial t} = -|\lambda_1|u + \varepsilon\Delta u - \frac{1}{2\mu}f_r(t,r),$$

$$\frac{\partial v}{\partial t} = -|\lambda_2|v + \varepsilon\Delta v + \frac{1}{2\mu}f_r(t,r). \tag{15}$$

These two equations can be easily solved to get the statistical moments of the random fields $u(t,r)$ and $v(t,r)$. For the large r for which $\Delta u = \partial^2 u/\partial r^2$, we can find from (15) (see Ref. [16]) that two-point equal time correlation functions are

$$\langle u(t,r)u(t,r')\rangle = \frac{D_\varphi}{4\mu^2(|\lambda_1|\varepsilon)^{1/2}} \exp\left[-\left(\frac{|\lambda_1|}{\varepsilon}\right)^{1/2}|r - r'|\right], \tag{16}$$

$$\langle v(t,r)v(t,r')\rangle = \frac{D_\varphi}{4\mu^2(|\lambda_2|\varepsilon)^{1/2}} \exp\left[-\left(\frac{|\lambda_2|}{\varepsilon}\right)^{1/2}|r - r'|\right]. \tag{17}$$

In what follows we consider only the case when

$$|\lambda_1| \ll |\lambda_2|. \tag{18}$$

One can see from (15) that under the condition (18) the random field $u(t,r)$ can be regarded as the "slow" field and $v(t,r)$ as the "fast" field. We can see from (16) and (17) that the ratio of the second moments $\langle v^2\rangle$ and $\langle u^2\rangle$ can be written as:

$$\frac{\langle v^2\rangle}{\langle u^2\rangle} = \left(\frac{|\lambda_1|}{|\lambda_2|}\right)^{1/2} \ll 1. \tag{19}$$

This inequality allows us to neglect the random fluctuations of $v(t,r)$ compared to those of $u(t,r)$.

4. Stochastic equation for order parameter

4.1. Adiabatic elimination

Our purpose now is to derive the stochastic equation governing the slow evolution of the field u [15]. Under the conditions (18) and (19) the "fast" field $v(t,r)$ follows the "slow" field $u(t,r)$. Neglecting partial structure of $v(t,r)$ one can find from (13) that

$$v = \frac{\delta(1-\varphi_\alpha(u))u}{2\mu\varepsilon\varphi_\beta(u) + \delta\varphi_\alpha(u) + g\mu^2}. \tag{20}$$

The latter equation corresponds to the so-called "silence" adiabatic elimination [15]. Substitution of (20) into (13) and putting $v = 0$ in (14) give the stochastic partial differential equation for the order parameter $u(t,r)$

$$\frac{\partial u}{\partial t} = b(u) + \varepsilon\nabla(\varphi_\beta(u)\nabla u) - \frac{1}{2\mu} f_r(t,r), \tag{21}$$

where

$$b(u) = \frac{2\mu(g\delta\varphi_\alpha(u) - \varepsilon^2\varphi_\beta^2(u))u}{2\mu\varepsilon\varphi_\beta(u) + \delta\varphi_\alpha(u) + g\mu^2}. \tag{22}$$

The main idea of this paper is that the stochastic reaction–diffusion equation (21) provides the universal description of magnetic field generation near *subcritical* bifurcation point. It admits a large variety of solutions including propagating fronts connecting the different metastable states. The remarkable result here is the appearance of the deterministic potential

$$U(u) = -\int_0^u b(z)\,\mathrm{d}z, \tag{23}$$

which one does not obtain by considering the original equations (3). Recall that the azimuthal component of the magnetic field, B_φ, can be found as $B_\varphi = v + u$. The function u describes how the solution of the system (3), (B_r, B_φ), moves along the eigenvector $\mathbf{h}_1$. Eliminating the variable v we neglect the "fast" evolution of (B_r, B_φ) towards $\mathbf{h}_1$ from arbitrary initial conditions.

4.2. Steady uniform distributions

Let us find the steady uniform distributions for Eq. (21). By using (8), (22), and equating $b(u)$ to zero, we find the equation:

$$\frac{\varepsilon^2}{g\delta}(1+u^2)^2(1+k_\alpha u^2) - (1+(k_\beta+1)u^2)^2 = 0, \tag{24}$$

determining non-trivial stationary points for the deterministic equation $\mathrm{d}u/\mathrm{d}t = b(u)$. If we take $k_\alpha = k_\beta = 1$, then (24) can be rewritten as the equation

$$(1+u^2)^3 - g_b(1+2u^2)^2 = 0 \tag{25}$$

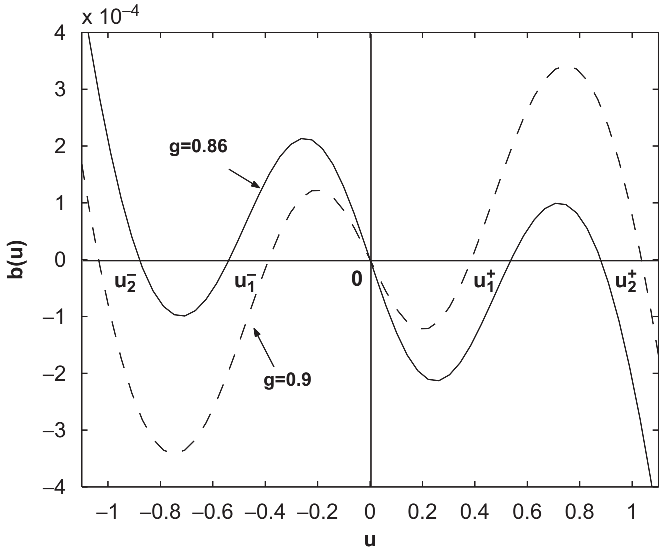

Fig. 1. Functions $b(u)$ of the stochastic PDE (21) for $\delta = 0.01$ and $\varepsilon = 0.1$ and two different values g. Solid line, $g = 0.86$; dashed line, $g = 0.9$.

with the bifurcation parameter

$$g_b = \frac{g\delta}{\varepsilon^2}. \tag{26}$$

It follows from (25) that when the parameter g_b is below approximately 0.844, there exists only one stable equilibrium point $u = 0$. For the range $0.844 < g_b < 1$, the system exhibits multistability in the bifurcation diagram [11]: there are two unstable states, $u_1^{\pm}$, and two non-trivial stable states, $u_2^{\pm}$ (see Fig. 1). This is a classical *subcritical* pitchfork bifurcation [17]. For example, if $g_b = 0.95$, then $u_1^{\pm} = \pm 0.243$ and $u_2^{\pm} = \pm 1.167$. The value $g_b = 1$ separates *subcritical* ($g_b < 1$) and *supercritical* ($g_b > 1$) zones. Here we are concerned only with the *subcritical* case: $0.844 < g_b < 1$. Since

$$\frac{|\lambda_1|}{|\lambda_2|} = \frac{\sqrt{g\delta} - \varepsilon}{\sqrt{g\delta} + \varepsilon} = \frac{1 - g_b}{(1 + \sqrt{g_b})^2} \ll 1, \tag{27}$$

the main criteria for the adiabatic elimination procedure: $|\lambda_1| \ll |\lambda_2|$ is generic in the *subcritical* case.

4.3. Free energy functional and first-order phase transitions

One can introduce the free energy functional

$$F[u] = \int \left[\frac{\varepsilon \varphi_\beta(u)}{2} (\nabla u)^2 + U(u) \right] \mathrm{d}r \tag{28}$$

such that the stochastic PDE (21) can be rewritten in the form

$$\frac{\partial u}{\partial t} = \frac{\delta F}{\delta u} - \frac{1}{2\mu} f_r(t, r). \tag{29}$$

This equation allows us to look at the problem of a galactic magnetic field generation as a first-order phase transition in a distributed non-equilibrium system [16]. The additive noise term, $f_r(t, r)$, represents the stochastic forcing arising from the small-scale fluctuations in magnetic and turbulent velocity fields. Here we address the situation when these fluctuations generate the critical nucleus. The stable uniform distributions $u = 0$ and $u = u_2^{\pm}$ can be interpreted as phases. *Subcritical* instability of the metastable state $u = 0$ with respect to a finite spatially localized perturbation (nucleus) gives rise to a transient behavior of magnetic field in the forms of trigger waves. They connect, for example, the local minimum of $F[u]$ at $u = 0$ and the global minimum at $u = u_2^{\pm}$. The critical nucleus, $u^*(t, \mathbf{r})$ can be found from $\delta F/\delta u = 0$, that is

$$b(u^*) + \varepsilon\nabla(\varphi_\beta(u^*)\nabla u^*) = 0, \tag{30}$$

$$\nabla u^*(0) = 0 \quad \text{and} \quad u^* \to 0 \quad \text{as } r \to \infty. \tag{31}$$

As long as the critical nucleus u^* is formed as a result of random perturbations, it gives rise to traveling fronts. One can verify that the functional Fokker–Planck equation corresponding to (29) has a stationary solution [17]

$$P[u] = \frac{1}{Z}\exp\left[-\frac{4\mu^2 F[u]}{D_\varphi}\right], \tag{32}$$

where Z is a normalization constant. The transition time, T, from the metastable uniform state $u = 0$ to the stable state u_2^+ is given by

$$T \sim \exp\left[\frac{4\mu^2 F[u^*]}{D_\varphi}\right]. \tag{33}$$

The derivation of this formula can be found in Ref. [22].

4.4. Magnetic fronts

In the *subcritical* case, galactic magnetic fronts can be analyzed by using the deterministic PDE

$$\frac{\partial u}{\partial t} = b(u) + \varepsilon\nabla(\varphi_\beta(u)\nabla u), \tag{34}$$

where the nonlinear function $b(u)$ belongs to a generic class of bistable nonlinearities: $b(u) < 0$ for u in $(0, u_1^+)$ and $b(u) > 0$ for u in (u_1^+, u_2^+). (Note that $b(u)$ is the odd function.) This is a classical reaction–diffusion equation with the field dependent diffusivity $\varepsilon\varphi_\beta(u)$ [7,23].

Let us consider the propagation of the effectively plane magnetic front neglecting all curvature effects. Eq. (34) can be written as

$$\frac{\partial u}{\partial t} = b(u) + \varepsilon\frac{\partial}{\partial r}\left(\varphi_\beta(u)\frac{\partial u}{\partial r}\right). \tag{35}$$

One can expect that the long-time development leads to the propagation of a traveling front of permanent form $u = u(z)$, where $z = r - ct$. The propagation rate c has to be found from the boundary value problem

$$-c\frac{du}{dz} = b(u) + \varepsilon\frac{d}{dz}\left(\varphi_\beta(u)\frac{du}{dz}\right), \tag{36}$$

$$u \to 0 \quad \text{as } z \to \infty, \quad u \to u_2^+ \quad \text{as } z \to -\infty. \tag{37}$$

One can also consider the front propagation when $u \to u_2^-$ as $z \to -\infty$. It should be noted that when the nonlinear function $b(u)$ is bistable and the diffusivity $\varepsilon\varphi_\beta(u)$ depends on the field u, the direction of front propagation is controlled by the sign of the integral

$$\int_0^{u_2^+} \varphi_\beta(u) b(u)\, du \tag{38}$$

rather than $\int_0^{u_2^+} b(u)\, du$ (constant diffusion) [7]. To show this, let us use the new variable [23] $s = s(z)$ obtained from the equation

$$\frac{ds}{dz} = \frac{1}{\varphi_\beta(u(z))}. \tag{39}$$

We can rewrite the boundary value problem (36) in terms of the auxiliary function $\Phi(s(z)) = u(z)$:

$$-c\,\frac{d\Phi}{ds} = \varphi_\beta(\Phi)b(\Phi) + \varepsilon\,\frac{d^2\Phi}{ds^2}, \tag{40}$$

$$\Phi \to 0 \quad \text{as } s \to \infty, \quad \Phi \to u_2^+ \quad \text{as } s \to -\infty. \tag{41}$$

Multiplying both sides of (40) by $d\Phi/ds$ and integrating over $[0, u_2^+]$, we find that the propagation rate c is given by

$$c = \frac{\int_0^{u_2^+} \varphi_\beta(u)b(u)\,du}{\int_{-\infty}^{\infty}\left(\frac{d\Phi}{ds}\right)^2 ds}. \tag{42}$$

So the speed c is positive as long as the integral (38) is positive. Fig. 2 shows the bistable function $b(u)$ $(u>0)$ (solid line) and the product $\varphi_\beta(u)b(u)$ (dashed line) appearing in the integral (38). It is clear from Fig. 2 and the formula (42) that the dependence of φ_β on u slows down the propagation speed.

To get the formula for c we approximate the nonlinear function $\varphi_\beta(u)b(u)$ by a cubic polynomial for $u\geqslant 0$:

$$\varphi_\beta(u)b(u) = \frac{|\lambda_1|u}{u_1^+u_2^+}(u_1^+ - u)(u - u_2^+), \tag{43}$$

such that $\varphi_\beta(0)b(0) = -|\lambda_1|u$. It follows from (40), (41) that the propagating rate c is given by

$$c = \sqrt{\frac{\varepsilon|\lambda_1|}{2u_1^+u_2^+}}(u_2^+ - 2u_1^+). \tag{44}$$

This is a unique speed which does not depend on the initial conditions. Recall that for the supercritical case when $\lambda_1>0$ we have the minimal speed $c_{\min} = 2\sqrt{\varepsilon\lambda_1}$.

So far we have considered the magnetic field propagation in the form of one-dimensional traveling wave that propagates with a constant velocity. It is well known that the speed of traveling wave in the two-dimensional depends on the radius of the expanding circle: $c(R) = c - D/R$, where D is a constant diffusivity, c is the propagation rate of a plane wave. So the excited domain with the radius less than the critical $R_{cr} = D/c$ does not propagate outward. The value of the critical radius corresponding to the equation with field

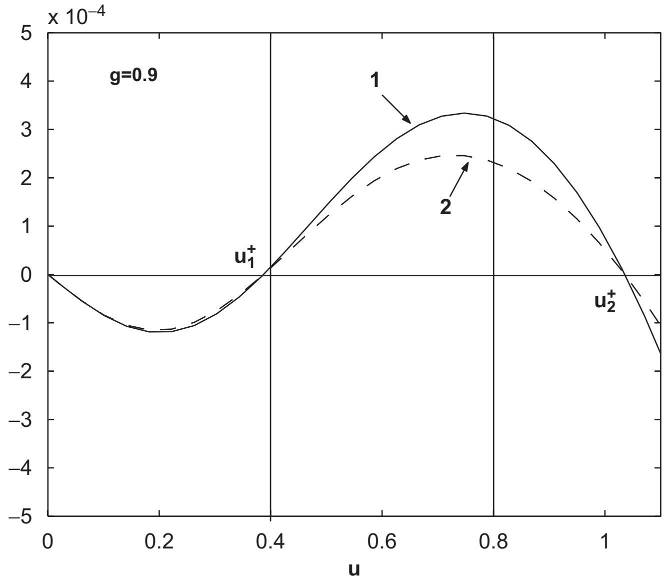

Fig. 2. Functions $b(u)$ (1—solid line) and $\varphi_\beta(u)b(u)$ (2—dashed line) for $g = 0.9$, $\delta = 0.01$ and $\varepsilon = 0.1$.

dependent diffusivity can be found from the critical "nucleus" problem (see (30) and (31)). Note that in the subcritical case, the excited domain with the radius greater than R_{cr} can be formed as a result of random fluctuations. Clearly, our results concerning the *subcritical* generation of magnetic field and front propagation are relevant not only for galactic dynamo but for solar dynamo as well where spatial and temporal structures emerge. However, the solar dynamo equations cannot be reduced to a single equation like (21) and therefore the computer simulations are required to analyze the *subcritical* dynamics.

5. Conclusions

We have studied the stochastic $\alpha\Omega$-dynamo model near the bifurcation point in the *subcritical* case. By using a two-component $\alpha\Omega$-dynamo model, we have derived the equation for order parameter in the form of a stochastic reaction–diffusion equation. This stochastic partial differential equation describes a *subcritical* generation of galactic magnetic field as a first-order phase transition in spatially extended system. We have identified the free energy functional for galactic dynamo problem which allows us to find the estimate for the mean transition time from the metastable uniform state with zero magnetic field. We have shown that the stochastic generation of magnetic field leads to a spontaneous front propagation. We have determined the speed of fronts which does not depend on the initial conditions.

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