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On the FKPP equation with Gaussian shear advection

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Abstract

A boundary value problem for the Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation with shear random advection is investigated. It is demonstrated that the upper bounds for front propagation in the FKPP equation with Gaussian advection obtained by an analysis of its linear ensemble-averaged upper solution can give too rough predictions. It is shown that the unboundedness of the Gaussian advection affects ensemble-averaged solutions of linear and nonlinear problems in a different way. This analytical prediction is confirmed by direct numerical simulations. The phenomenon of propagation failure due to zero boundary conditions is studied and the critical conditions are found. Numerical procedure for the analysis of random wave propagation is suggested by introduction of mean front position and its variance. Some numerical experiments are presented. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper is concerned with an asymptotic study of a boundary value problem for the Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation with random advection [1–4] (see also [5–8]). Our original motivation came from the application to turbulent combustion where the main problem is to describe the enhancement of the total combustion rate in turbulent flow [9]. A great deal of progress in this area has recently been made by applying the FKPP equation with turbulent convection describing the turbulent reaction front propagation [1–4]. The key question when using FKPP equation to model turbulent combustion is to determine the speed of the reaction front as a function of the statistical characteristics of turbulent velocity field. Another objective is to derive the equations governing a large-scale evolution of reaction-transport process. The FKPP equation is also widely applied for description of various phenomena in biological sciences (see [10–12] and references therein).

Some upper bounds for front propagation in the FKPP equation with Gaussian random advection have been obtained in [1–3]. In [2] the authors raised the question how accurate the ensemble-averaged bounds are. It was

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supposed that in some cases the ensemble-averaged upper bounds are extremely pessimistic compared with the upper bounds for almost every realization of the random advection. The main feature of these works has been the use of the linearization procedure for FKPP equation. In this contest the natural question arises whether or not we can use the linear approximation for FKPP equation with unbounded random advection to get accurate estimates for the front propagation speed. It should be noted that in the situation when the coefficients of the FKPP equation are bounded, a solution to the linearized problem provides quite accurate upper bounds for the solution of the original nonlinear problem (see, e.g. [5,6]).

It is the purpose of this work to show that consideration of a linear parabolic equation instead of the full nonlinear problem with Gaussian random advection can lead to inaccurate results. By using a simple model we show that the analysis of the ensemble-averaged *upper* solution (i.e. solution to the linearized problem) can give too rough (even in an asymptotic sense) upper bounds for the ensemble-averaged solution of the FKPP equation with Gaussian random advection. The reason for this can be explained by the fact that unboundedness of the advection affects ensemble-averaged solutions of linear and nonlinear problems in a different way. We confirm our analytical predictions by direct numerical simulations. We conclude that the problem of obtaining fairly accurate upper bounds for the FKPP equation with Gaussian random advection cannot be solved, in general, by simple linear analysis. The full nonlinear problem becomes extremely difficult from the analytical point of view and therefore numerical tools are needed.

Further, the ensemble-averaged solution alone gives us a limited description of the front propagation. Indeed, knowing upper bounds for the ensemble-averaged solution, we are able to indicate where the solution of original problem goes almost surely (a.s.) to zero as time goes to infinity. But there are other questions concerning front propagation which cannot be described by the ensemble-averaged solution. Majda and Souganidis [4] proposed a method to find a random front position by solving a variational problem with randomness. However, to find the solutions to this variational problem in a constructive way is an extremely difficult task.

Here, to give an adequate description of the wave propagation for FKPP equation with random advection, we suggest the numerical procedure by introducing the mean front position and its variance. Clearly, numerical evaluation of these characteristics is not a more complicated task than simulation of the ensemble-averaged solution of the nonlinear problem.

In this paper, we consider the Dirichlet problem for the FKPP equation with random shear advection in a band (the FKPP equation with random shear advection in plane was studied in [1,2,4] and the deterministic FKPP equation in a band was studied in [5,6]). The zero boundary condition might lead to the phenomenon of the propagation failure. It happens when the normalized diffusion coefficient across the band is large in comparison to the reaction rate (cf. [5]).

2. Ensemble-averaged solution

Consider the FKPP equation with random convection in a band

$$\frac{\partial u}{\partial t} = \frac{D_x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{D_y}{2}\frac{\partial^2 u}{\partial y^2} + V(y)\frac{\partial u}{\partial x} + cu(1-u), \quad t > 0, \quad x \in R, \quad y \in (-l, l), \tag{1}$$

with the initial and boundary conditions

$$u(0, x, y) = u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{2}$$

$$u(t, x, \pm l) = 0, \tag{3}$$

where the diffusion coefficients D_x and D_y and the reaction rate parameter c are positive constants and $V(y)$ is a Gaussian random field with zero mean defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$.

Introduce the correlation function for the random field $V(y)$

$$K(y_1, y_2) = \hat{E}V(y_1)V(y_2).$$

Here and below $\hat{E}$ denotes mathematical expectation over the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. We shall suppose that almost every realization of $V(y)$ is sufficiently smooth (see sufficient conditions for this in, e.g. [13]) and that for all $y \in [-l, l]$

$$K(y, y) = \hat{E}(V(y))^2 \leq K_0. \tag{4}$$

Eq. (1) is one of the simplest models incorporating the combined effects of diffusion, nonlinearity, and random advection. Previously, such an equation was studied in plane [1,2,4]. In [1,2] some upper bounds for the ensemble-averaged solution were obtained. And the problem of (random) wave propagation was reduced to consideration of a variational problem with randomness in [4]. The boundary condition (3) may lead to an additional effect of propagation failure when there is no propagation of the wave. It happens when the normalized diffusion coefficient across the band D_y/l^2 is greater than the reaction rate c (cf. [5], Section 6.3).

We start with the estimation for an ensemble-averaged *upper* solution. To this end, consider the linearized problem

$$\frac{\partial v}{\partial t} = \frac{D_x}{2}\frac{\partial^2 v}{\partial x^2} + \frac{D_y}{2}\frac{\partial^2 v}{\partial y^2} + V(y)\frac{\partial v}{\partial x} + cv, \quad t > 0, \quad x \in R, \quad y \in (-l, l), \tag{5}$$

$$v(0, x, y) = u_0(x), \tag{6}$$

$$v(t, x, \pm l) = 0. \tag{7}$$

It is well known (see, e.g. [5,6]) that the solution to this problem is an upper one for (1)–(3), i.e.

$$u(t, x, y) \leq v(t, x, y), \quad t \geq 0, \quad x \in R, \quad y \in [-l, l].$$

To evaluate the ensemble-averaged upper solution

$$\bar{v}(t, x, y) := \hat{E}v(t, x, y),$$

we use techniques developed in [2,5].

The Fourier transform of v

$$w(t, y; \xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi}v(t, x, y)\,\mathrm{d}x,$$

satisfies the problem

$$\frac{\partial w}{\partial t} = \frac{D_y}{2}\frac{\partial^2 w}{\partial y^2} + \left(-\frac{D_x}{2}\xi^2 + \mathrm{i}\xi V(y) + c\right)w, \quad t > 0, \quad y \in (-l, l), \tag{8}$$

$$w(0, y; \xi) = w_0(\xi), \tag{9}$$

$$w(t, \pm l; \xi) = 0, \tag{10}$$

where $w_0(\xi)$ is the Fourier transform of $u_0(x)$.

Using probabilistic representations of solutions to linear parabolic problems (see, e.g. [5,14]), we get

$$w(t,y;\xi)=w_0(\xi)\exp\left(ct-\frac{D_x}{2}\xi^2t\right)E\left[\chi_{\tau_y>t}\exp\left(\mathrm{i}\xi\int_0^t V(y+\sqrt{D_y}W(s))\,\mathrm{d}s\right)\right], \tag{11}$$

where $W(s)$, $s\geq 0$, is a one-dimensional standard Wiener process, τ_y is the first exit time of the process $Y_y(s)=y+\sqrt{D_y}W(s)$, $s\geq 0$, $y\in[-l,l]$, from the interval $[-l,l]$, E denotes the mathematical expectation over the probability space on which the Wiener process $W(s)$ is defined, and χ_A is an indicator of the set A. Let us note that $V(y)$ and $W(s)$ are independent.

From Eq. (11) we obtain

$$\bar{w}(t,y;\xi):=\hat{E}w(t,y;\xi)=w_0(\xi)\exp\left(ct-\frac{D_x}{2}\xi^2t\right)E\left[\chi_{\tau_y>t}\hat{E}\exp\left(\mathrm{i}\xi\int_0^t V(y+\sqrt{D_y}W(s))\,\mathrm{d}s\right)\right].$$

Using the well-known property of Gaussian random fields, we get

$$\bar{w}=w_0(\xi)\exp\left(ct-\frac{D_x}{2}\xi^2t\right)E\left[\chi_{\tau_y>t}\exp\left(-\frac{1}{2}\xi^2I\right)\right], \tag{12}$$

where

$$I=I(t,y,W(\cdot)):=\hat{E}\left[\int_0^t V(y+\sqrt{D_y}W(s))\,\mathrm{d}s\right]^2.$$

Note that $I\geq 0$ is random and

$$\begin{aligned}I&=\int_0^t\int_0^t \hat{E}V(y+\sqrt{D_y}W(s'))V(y+\sqrt{D_y}W(s''))\,\mathrm{d}s'\,\mathrm{d}s''\\&=\int_0^t\int_0^t K(y+\sqrt{D_y}W(s'),y+\sqrt{D_y}W(s''))\,\mathrm{d}s'\,\mathrm{d}s''.\end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}\bar{v}(t,x,y)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}u_0(\zeta)\int_{-\infty}^{\infty}\exp(-\mathrm{i}\xi(x-\zeta))\exp\left(ct-\frac{D_x}{2}\xi^2t\right)E\left[\chi_{\tau_y>t}\exp\left(-\frac{1}{2}\xi^2I\right)\right]\mathrm{d}\xi\,\mathrm{d}\zeta\\&=\frac{1}{2\pi}\mathrm{e}^{ct}E\left[\chi_{\tau_y>t}\int_{-\infty}^{0}\left(\int_{-\infty}^{\infty}\exp(-\mathrm{i}\xi(x-\zeta)-\frac{1}{2}\xi^2(D_xt+I))\,\mathrm{d}\xi\right)\mathrm{d}\zeta\right].\end{aligned}$$

By introducing new variables

$$\eta=\xi\sqrt{2(D_xt+I)},\qquad z=\frac{x-\zeta}{\sqrt{2(D_xt+I)}},$$

we can get

$$\begin{aligned}\bar{v}(t,x,y)&=\frac{1}{2\pi}\mathrm{e}^{ct}E\left[\chi_{\tau_y>t}\int_{x/\sqrt{2(D_xt+I)}}^{\infty}\left(\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\eta z-\frac{\eta^2}{4}\right)\mathrm{d}\eta\right)\mathrm{d}z\right]\\&=\frac{1}{\sqrt{\pi}}\mathrm{e}^{ct}E\left[\chi_{\tau_y>t}\int_{x/\sqrt{2(D_xt+I)}}^{\infty}\mathrm{e}^{-z^2}\,\mathrm{d}z\right].\end{aligned} \tag{13}$$

It follows that

$$I\leq K_0t^2, \tag{14}$$

where K_0 is from (4). The relations (13) and (14) imply that for $x > 0$

$$\bar{v}(t,x,y) \le \hat{v}(t,x,y) := \frac{1}{\sqrt{\pi}} \mathrm{e}^{ct} \int_{x/\sqrt{2(D_x t + K_0 t^2)}}^{\infty} \mathrm{e}^{-z^2}\, \mathrm{d}z\, E\chi_{\tau_y > t}. \tag{15}$$

And it is evident that for $x \le 0$

$$\bar{v}(t,x,y) \le \mathrm{e}^{ct} E\chi_{\tau_y > t}.$$

We introduce the function

$$\psi(t,y) := E\chi_{\tau_y > t} = P(\tau_y > t).$$

This function satisfies the problem [5,14]

$$\frac{\partial \psi}{\partial t} = \frac{D_y}{2} \frac{\partial^2 \psi}{\partial y^2}, \quad t > 0, \quad y \in (-l, l),$$

$$\psi(0,y) = 1, \qquad \psi(t, \pm l) = 0.$$

The solution of this problem can be written as

$$\psi(t,y) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{\pi(2k+1)y}{2l}\right) \exp\left(-\frac{\pi^2 D_y}{8l^2}(2k+1)^2 t\right).$$

Then for $y \in (-l, l)$

$$\psi(t,y) \sim \exp\left(-\frac{\pi^2 D_y}{8l^2} t\right) \text{ as } t \to \infty. \tag{16}$$

Whence we get

$$\text{for } x > 0, \quad y \in (-l,l): \quad \hat{v} \sim \exp\left(ct - \frac{x^2}{2(D_x t + K_0 t^2)} - \frac{\pi^2 D_y}{8l^2} t\right) \text{ as } t \to \infty, \tag{17}$$

$$\text{for } x \le 0, \quad y \in (-l,l): \quad \hat{v} \sim \exp\left(ct - \frac{\pi^2 D_y}{8l^2} t\right) \text{ as } t \to \infty. \tag{18}$$

Finally, we come to the following proposition (cf. [5, p. 435]).

Proposition 1. *If* $c < \pi^2 D_y / 8l^2$, *then* $\lim_{t\to\infty} \bar{u}(t,x,y) = 0$, *where* $\bar{u}(t,x,y)$ *is the ensemble-averaged solution of the problem* (1)–(3).

It is clear that

$$0 \le u(t,x,y) \le 1.$$

Due to this fact and the above proposition, we get the following assertion.

Proposition 2. *If* $c < \pi^2 D_y / 8l^2$, *then* $\lim_{t\to\infty} u(t,x,y) = 0$, $\hat{P}$-*a.s.*, *where* $u(t,x,y)$ *is the solution of the problem* (1)–(3).

Thus, there is no propagation of the wave, when the normalized diffusion coefficient across the band $\pi^2 D_y/8l^2$ is greater than the reaction rate c.

Now consider the case of $c > \pi^2 D_y/8l^2$. It follows from Eq. (17) that

$$\text{if} \quad c > \frac{\pi^2 D_y}{8l^2}, \tag{19}$$

then

$$\lim_{t\to\infty} \bar{v}(t, Ct^{3/2}, y) = 0,$$

with $C > \sqrt{2K_0(c - \pi^2 D_y/8l^2)}$ and, consequently,

$$\lim_{t\to\infty} \bar{u}(t, Ct^{3/2}, y) = 0. \tag{20}$$

One can conclude from (19), (20) that for $c > \pi^2 D_y/8l^2$ the front of the ensemble-averaged solution $\bar{u}$ accelerates (cf. [2]). But in fact the bound (19), (20), which is a result of the linear analysis, is too rough as an upper bound of the solution to the nonlinear problem (1)–(3). In fact, there is no front acceleration for nonlinear problem. Below we give two illustrative examples approving this. As we shall see in Example 1, the reason of the roughness of the bound (19), (20) is not in an inaccurate estimation of the upper solution $\bar{v}$ by $\hat{v}$ in (15) but in the fact that unboundedness of the advection affects ensemble-averaged solutions of the linear and nonlinear problems in a different way. Direct numerical simulations of Example 2 also clearly indicate that there is no acceleration of the front here (see Fig. 3).

Let us note that when the coefficients are bounded, a solution to the linearized problem provides quite accurate upper bounds for the solution of a FKPP equation (see, e.g. [5,6]). In particular, it is proved in [5, p. 435] that if $c > \pi^2 D_y/8l^2$ then the front in the problem (1)–(3) without advection (i.e. when $V(y) = 0$) propagates with the velocity $\sqrt{2D_x(c - \pi^2 D_y/8l^2)}$ for $t \gg 1$. And one can see that for $V(y) = 0$ the front of the upper solution propagates asymptotically with the same velocity (see Eq. (17)).

Example 1. To demonstrate that the bound (19), (20) is too rough, we consider the simplest case of $V(y)$, when it is a random constant:

$$V(y) \equiv V,$$

where V is a Gaussian random variable with zero mean and variance σ^2.

For simplicity, let us also put $l = \infty$. Then, the problem (1)–(3) is reduced to the one-dimensional Cauchy problem (we use here the same letter u but it will not cause any confusion):

$$\frac{\partial u}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2} + V\frac{\partial u}{\partial x} + cu(1-u), \quad t > 0, \quad x \in R, \tag{21}$$

$$u(0, x) = u_0(x), \tag{22}$$

where $u_0(x)$ is the same as in (2).

Let us denote by $v(t, x)$ the upper solution for (21), (22), which satisfies the linear problem (cf. (5), (6))

$$\frac{\partial v}{\partial t} = \frac{D}{2}\frac{\partial^2 v}{\partial x^2} + V\frac{\partial v}{\partial x} + cv, \quad t > 0, \quad x \in R, \tag{23}$$

$$v(0, x) = v_0(x), \tag{24}$$

Since $V(y) = V$, the inequality (14) (and therefore (15)) becomes the exact equality. Then the ensemble-averaged upper solution $\bar{v}(t, x) := \hat{E}v(t, x)$ is equal to (cf. (15))

$$\bar{v}(t, x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{ct} \int_{x/\sqrt{2(D_x t+\sigma^2 t^2)}}^{\infty} \mathrm{e}^{-z^2}\,\mathrm{d}z.$$

It follows from here that

$$\lim_{t\to\infty} \frac{1}{t} \ln \bar{v}(t, Ct^{3/2}) = c - \frac{C^2}{2\sigma^2} \quad \text{for} \quad 0 < C < \sqrt{2c\sigma^2} \tag{25}$$

and

$$\lim_{t\to\infty} \bar{v}(t, Ct^{3/2}) = 0 \quad \text{for } C > \sqrt{2c\sigma^2}. \tag{26}$$

We see that the front of the domain $\{x > 0 : \bar{v}(t, x) > 0\}$ accelerates as $t \to \infty$.

Let us explain this acceleration in the linear problem (23), (24) by qualitative arguments. Due to the well-known results (see, e.g. [5,6]), for every $-\infty < V < \infty$ and $t \gg 1$ the real line $\boldsymbol{R}$ is separated at $\approx\sqrt{2Dc}t - Vt$ into two sets: a set where $v(t, x) \sim 1$ or greater and its complement where $v(t, x)$ is close to 0. Hence, $v(t, Ct^{3/2})$ can be of order 1 when the random variable V takes values less than or equal to $-Ct^{1/2}$. The probability

$$\hat{P}(V \le -Ct^{1/2}) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{-Ct^{1/2}} \mathrm{e}^{-z^2/2\sigma^2}\,\mathrm{d}z \sim \mathrm{e}^{-C^2 t/2\sigma^2} \text{ as } t \to \infty.$$

Therefore, the probability

$$\hat{P}(v(t, Ct^{3/2}) \sim 1) \sim \mathrm{e}^{-C^2 t/2\sigma^2} \text{ as } t \to \infty. \tag{27}$$

In the linear model (23), (24) the solution $v(t, x)$ can grow infinitely as e^{ct}. Then, the realizations of v corresponding to Eq. (27) give contribution $\sim\mathrm{e}^{ct-C^2 t/2\sigma^2}$ to the mean $\bar{v}(t, Ct^{3/2})$. As a result, we come to the conclusion that the front accelerates (see (25), (26)) for the ensemble-averaged solution $\bar{v}$ of the linear problem (23), (24). Let us note that here, in fact, Gaussian advection induces diffusion, which is a usual effect in linear parabolic equations with Gaussian advection (see, e.g. [15] and references therein).

Now consider the nonlinear problem (21), (22). In comparison to the solution of (23), (24), which can grow infinitely, the solution $u(t, x)$ of the nonlinear problem (21), (22) is always bounded: $0 \le u \le 1$. This is a crucial point which causes an essential difference in behavior of ensemble-averaged solutions of the linear and nonlinear problems.

Let $x^*(t)$ be such that

$$u(t, x^*(t)) = \tfrac{1}{2}. \tag{28}$$

Using the well-known results for the FKPP equation [16], we obtain that for every $-\infty < V < \infty, x^*(t) \approx \sqrt{2Dc}t - Vt$ under $t \gg 1$. Since $\hat{P}(V < 0) = \hat{P}(V > 0) = 1/2$, we get that for $t \gg 1$, $u(t, \sqrt{2Dc}t) \sim 1$ with probability $1/2$ and $u(t, \sqrt{2Dc}t) \sim 0$ with probability $1/2$. Therefore, the ensemble-averaged solution $\bar{u}(t, \sqrt{2Dc}t) = \hat{E}u(t, \sqrt{2Dc}t) \approx 1/2$ for big t. Further, it is clear that (recall that $0 \le u \le 1$) for $t \gg 1$

$$\bar{u}(t, \sqrt{2Dc}t + \alpha t) \approx \hat{P}(V < -\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\alpha}^{\infty} \mathrm{e}^{-z^2/2\sigma^2}\,\mathrm{d}z,$$

whence for any $\alpha(t) \to \infty$ as $t \to \infty$ we obtain (cf. (26))

$$\bar{u}(t, \sqrt{2Dc}\,t + \alpha(t)t) \to 0 \text{ as } t \to \infty. \tag{29}$$

So, for $t \gg 1$ the front of the ensemble-averaged solution propagates with the velocity $\sqrt{2Dc}$, we have no acceleration of the front in the nonlinear problem (21), (22).

This example explains why the ensemble-averaged upper bounds obtained by the linear analysis can be extremely pessimistic in the case of the FKPP equation with random Gaussian advection, which can take arbitrary large values. (Note that in [1–3] the upper bounds for solutions of such a kind of FKPP equations was obtained by an analysis of their ensemble-averaged upper solutions.) Thus, to get an accurate upper bound for the ensemble-averaged solution of the FKPP equation with Gaussian advection, we should average a solution to the nonlinear equation. In general case it is extremely difficult problem to work with. In our opinion numerical methods should be applied in this case. Further, the ensemble-averaged solution can give us only the limited description of the front propagation. Indeed, knowing the bounds like (29), we are able to indicate, where the solution of the original problem goes a.s. to 0 as t goes to infinity. However, there are other questions on front propagation, in which one can also be interested (e.g. various statistical characteristics of front position and front velocity) but which are not touched by the ensemble-averaged solution. In the next section we suggest the numerical procedure by introducing the mean front position and its variance.

3. Mean front position and its variance

It is possible to prove the following assertion (see [5, p. 435]).

Proposition 3. *For $c > \pi^2 D_y/8l^2$*

$$\lim_{t\to\infty} u(t,x,y) \le u^*, \quad \hat{P}\text{-}a.s.,$$

where $u(t,x,y)$ is the solution of the problem (1)–(3) *and $u^* = 1 - \pi^2 D_y/8l^2 c$.*

Note that the constant u^* is the precise upper bound for the limit in Proposition 3.

Definition 1. Let $x^*(t)$ be the (random) position of the wave front for the problem (1)–(3), which is defined as the largest point x with $u(t,x,y) = u^*/2$.

Introduce the characteristics for description of wave propagation in the model (1)–(3): the mean position of the wave front

$$\bar{x}^*(t) := \hat{E}x^*(t),$$

and its variance

$$Dx^*(t) := \hat{E}\left(x^*(t) - \bar{x}^*(t)\right)^2.$$

From numerical point of view, evaluation of these characteristics is not a more complicated problem than simulation of an ensemble-averaged solution of the nonlinear equation. And the characteristics $\bar{x}^*(t)$, $Dx^*(t)$ give us quite adequate description of front propagation in the model (1)–(3).

Let us note that due to the Chebyshev inequality the (random) position of the front $x^*(t)$ is located at

$$x^*(t) = \bar{x}^*(t) \pm \gamma\sqrt{Dx^*(t)},$$

with probability at least not less than $(\gamma^2 - 1)/\gamma^2$, $\gamma > 1$.

Remark 1. Return to the model (21), (22) from Example 1 and to the random function $x^*(t)$ introduced in Eq. (28). We have noted above that for every $-\infty < V < \infty$ the front position $x^*(t) \approx \sqrt{2Dc}t - Vt$ at $t \gg 1$. Since V in Example 1 is a Gaussian random variable with 0 mean and variance σ^2, the position of the front $x^*(t)$ is located at $\sqrt{2Dc}t \pm 3\sigma t$ with probability 0.997. We see that for $t \gg 1$ the mean position of the front $\bar{x}^*(t) \approx \sqrt{2Dc}t$, the standard deviation $\sqrt{Dx^*(t)} \approx \sigma t$, and the mean velocity $\bar{v} = \bar{x}^*(t)/t \approx \sqrt{2Dc}$. Note that both the mean position and the standard deviation increase linearly with time (see Example 2 as well). In a general case it is possible to find the magnitudes $\bar{x}^*(t)$, $Dx^*(t)$ numerically.

Since we consider front propagation in the band, it is natural to assume that the random advection $V(y)$ (i.e. the velocity of random flow in which the wave propagation takes place) is equal to zero on the band boundary:

$$V(\pm l) = 0.$$

Then, the random function $V(y)$ can be presented as (see, e.g. [17], Section 1.7)

$$V(y) = \sum_{k=0}^{\infty}\left(a_k\xi_k \cos\left(\frac{\pi(2k+1)y}{2l}\right) + b_k\zeta_k \sin\left(\frac{\pi ky}{l}\right)\right), \tag{30}$$

where ξ_k and ζ_k are mutually independent normally distributed $N(0, 1)$ random variables with zero mean and unit standard deviation. And the correlation function $K(y_1, y_2)$ has the form

$$K(y_1, y_2) = \sum_{k=0}^{\infty}\left(a_k^2 \cos\left(\frac{\pi(2k+1)y_1}{2l}\right) \cos\left(\frac{\pi(2k+1)y_2}{2l}\right) + b_k^2 \sin\left(\frac{\pi ky_1}{l}\right) \sin\left(\frac{\pi ky_2}{l}\right)\right).$$

We will return to a detailed analysis of the front propagation in the model (1)–(3) with random advection $V(y)$ of the form (30) in a later publication. Besides, an important and interesting direction of further research is to investigate the problem with random advection, which has complex spatiotemporal statistics [18,19]. Here we will restrict ourselves to an illustrative example.

Example 2. Consider the simple case of (30):

$$V(y) = a\xi \cos\left(\frac{\pi(2k+1)y}{2l}\right), \tag{31}$$

where $a > 0$ is a constant, k is an integer, and ξ is a normally distributed $N(0, 1)$ random variable.

To illustrate the introduced characteristics, we present some numerical experiments. The numerical simulations have been conducted for the following set of parameters: $D_x = D_y = 0.5$, $l = 1$, $a = 2$, $k = 3$ and $c = 2.3$. In this case $c > \pi^2 D_y/8l^2$ and $u^* \approx 0.87$.

First let us give one of possible solutions of the problem (1)–(3). Fig. 1 demonstrates a solution to this problem without advection ($V(y) = 0$). We see that the front is quite sharp. Obviously, the solution has, in a sense, a similar form for other realizations of $V(y)$ from (31).

We use the Monte Carlo technique to get statistical characteristics for the problem (1)–(3) with $V(y)$ from Eq. (31). We simulate independent random variables ξ_k, $k = 1, \ldots, N$, distributed as ξ. And using layer methods of [20], we simulate N solutions of the considered problem, each solution corresponds to its ξ_k. The layer methods for solving nonlinear partial differential equations of parabolic type are constructed using a probabilistic approach [20,21]. Despite their probabilistic nature these methods are nevertheless deterministic. The probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an

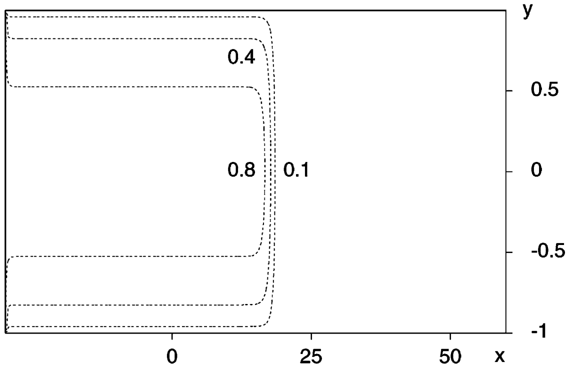

Fig. 1. Level curves of the solution $u(t, x, y)$ of the problem (1)–(3) with $V(y) = 0$ at time moment $t = 20$ for the parameters given in Example 2.

intrinsic manner. The layer methods allow us to avoid difficulties stemming from essentially changing coefficients and strong advection, which take place in the simulated problem.

In our experiments we take $N = 400$ and mainly use a first-order layer method from [20] with time step $h = 0.02$. We turn the layer method into a numerical algorithm by attracting cubic interpolation with space steps $h_x = 0.2$ and $h_y = 0.04$ in x and y directions correspondingly (see details in [20]). To control the accuracy, from time to time we make simulations with smaller steps and use linear interpolation instead of the cubic one. The algorithm has been realized on a moving mesh. The Monte Carlo error is not more than 5% of the results.

Fig. 2 gives an ensemble-averaged solution to the considered problem at time $t = 20$. The front width of the ensemble-averaged solution increases in time (see also Remark 1). One can say that $u(t, x, y) \sim 0$ for $t \leq 20$, $x > 60$ with high probability. But it is difficult to extract more information from an ensemble-averaged solution. The characteristics $\bar{x}^*(t)$, $Dx^*(t)$ provide us with an additional information on the behavior of the solution. Fig. 3 presents time dependence of the mean front position $\bar{x}^*(t)$ and its standard deviation $\sqrt{Dx^*(t)}$. We see that $\bar{x}^*(t)$ is a linear function in t for sufficiently large times: $\bar{x}^*(t) \approx \bar{v}t$, where $\bar{v}$ is the mean velocity. There is no front acceleration (cf. Section 2). The standard deviation is also a linearly increasing function in time. This explains, in particular, increase of front width of the ensemble-averaged solution in time (see Fig. 2 and also Remark 1).

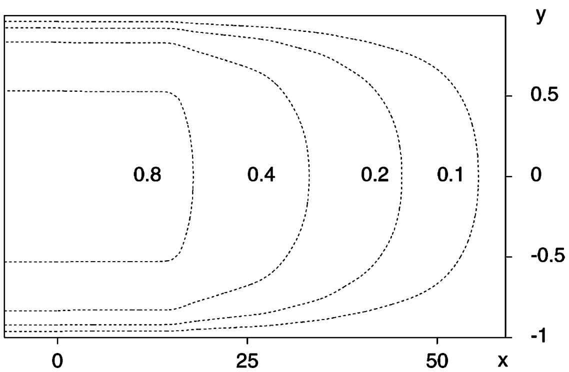

Fig. 2. Level curves of the ensemble-averaged solution $\bar{u}(t, x, y)$ of the problem (1)–(3) with $V(y)$ from (31) at $t = 20$ for the parameters given in Example 2.

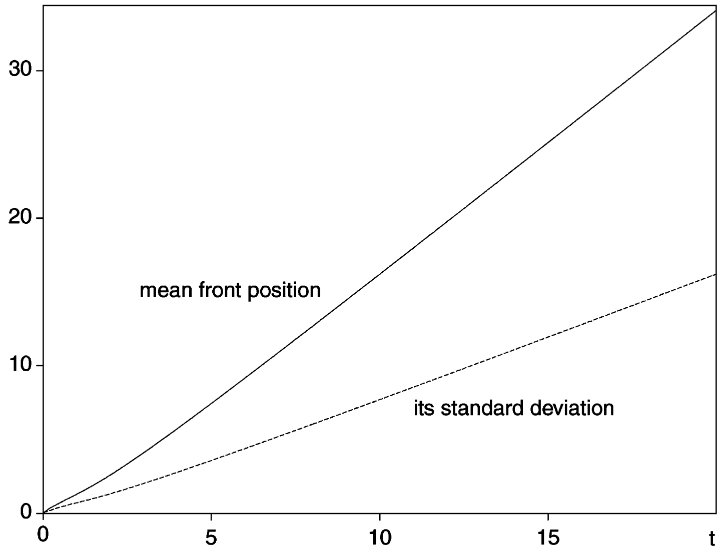

Fig. 3. The mean front position $\bar{x}^*(t)$ and its standard deviation $\sqrt{Dx^*(t)}$ for the parameters given in Example 2.

Let us note that for the same values of parameters as above but without advection (i.e. $V(y) = 0$) the asymptotic velocity of front propagation is equal to $\sqrt{2D_x(c - \pi^2 D_y/8l^2)} \approx 1$ (see also Fig. 1). In our numerical experiments, i.e. in the case of random advection (31), the mean velocity $\bar{v}$ is approximately equal to 1.8. So, the random advection with zero mean can lead to increase of the mean velocity of front propagation. Apparently, this is due to the fact that for each realization of ξ the drift $V(y)$ is positive for some y and thus there is, roughly speaking, an effective drift in the model.

4. Summary and conclusions

We have studied the long-time behavior of the solution of a boundary value problem for the FKPP equation with random shear flow. We have employed a probabilistic representation of solutions to parabolic problems to get the ensemble-averaged *upper* solutions. We have found that the latter give too rough (even in an asymptotic sense) upper bounds for the ensemble-averaged solution of the full nonlinear FKPP equation with Gaussian advection. It has been shown that the unboundedness of the advection affects ensemble-averaged solutions of linear and nonlinear problems in a different way. We have confirmed our analytical predictions by direct numerical simulations. We have also studied the phenomenon of propagation failure due to zero boundary conditions and found the conditions under which there is no propagation of waves.

We have suggested the numerical procedure for the analysis of wave propagation for FKPP equation by introducing the mean front position and its variance. The basic advantage of these characteristics is that they give an adequate description of the model's behavior while their numerical evaluation is not a more complicated task than simulation of the ensemble-averaged solution of the full nonlinear problem. We have presented some numerical results.

In this paper, we have considered only a simple model of shear advection that allows us to demonstrate some general features of solutions of the FKPP equation with random advection. An important and interesting direction of further research is to study front propagation for FKPP equation with turbulent advection, which has complex spatiotemporal statistics [18,19].

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