



Stochastic analysis of subcritical amplification of magnetic energy in a turbulent dynamo

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Abstract

We present and analyze a simplified stochastic $\alpha\Omega$ -dynamo model which is designed to assess the influence of additive and multiplicative noises, non-normality of dynamo equation, and nonlinearity of the α -effect and turbulent diffusivity, on the generation of a large-scale magnetic field in the subcritical case. Our model incorporates random fluctuations in the α -parameter and additive noise arising from the small-scale fluctuations of magnetic and turbulent velocity fields. We show that the noise effects along with non-normality can lead to the stochastic amplification of the magnetic field even in the subcritical case. The criteria for the stochastic instability during the early kinematic stage are established and the critical value for the intensity of multiplicative noise due to α -fluctuations is found. We obtain numerical solutions of nonlinear stochastic differential equations and find the series of phase transitions induced by random fluctuations in the α -parameter.

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1. Introduction

The understanding of the generation and maintenance of a large scale magnetic field in astrophysical objects is a problem of exceptional importance and difficulty.

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It is widely accepted that the magnetic field is generated by the turbulent flow of the electrically-conducting fluid. Inhomogeneous velocity fluctuations stretch magnetic lines and amplify the magnetic field. These small scale fluctuations of turbulent flow are primarily responsible for the generation of magnetic fields. The problem is that it is difficult to resolve them. Thus their influence on the resolved large-scale magnetic field has to be modelled. A traditional closure scheme is based on the α -effect according to which small-scale fluctuations can be described by an average term involving the curl of the mean magnetic field, \mathbf{B} , written as $\nabla \times (\alpha \mathbf{B})$. The asymptotic analysis of the induction equation exploiting the assumption of two separated scales for turbulent flow leads to the effective macroscopic equation for the large scale magnetic field $\mathbf{B}(t, \mathbf{x})$ [1–5].

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) - \nabla \times (\beta \nabla \times \mathbf{B}) + \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (1)$$

where \mathbf{u} is the mean velocity field of the turbulent flow, α is the coefficient of the α -effect and β is the turbulent magnetic diffusivity. Traditionally the phenomenon of generation of magnetic fields was analyzed by considering perturbations of the trivial state $\mathbf{B} = 0$ and looking for exponential solutions to the deterministic PDE (1) with appropriate boundary conditions. While this standard stability analysis successfully predicts the dynamo action for the supercritical case, there are situations for which this eigenvalue analysis fails to predict the subcritical onset of instability [6–10]. It was pointed out in Ref. [9] that the closure scheme involving only deterministic $\alpha\beta$ -parameterization is not completely satisfactory since unresolved fluctuations may produce random terms on the right-hand side of the dynamo equation (1). Moreover, it follows from astronomical observations that large scale magnetic fields exhibit a rich random variability both in space and time that cannot be described by the deterministic equation (1).

The importance of noise effects in the dynamo problem has been recognized previously and several attempts have been made to account for the effects of spatial and temporal fluctuations in small scale magnetic and velocity fields on the generation process. Kraichnan considered fluctuations in the α -parameter and found a negative contribution to turbulent diffusivity from helicity fluctuations [11]. Hoyng with colleagues in Refs. [12–14] studied in detail the effect of random fluctuations in the α -parameter by considering the system of stochastic linear equations for eigenmodes corresponding to the dynamo equations. They found the excitation of those modes such that their magnetic energy is proportional to γ^{-1} , where γ is the damping rate. Stochastic dynamics of magnetic field generation have been also analyzed by Farrell and Ioannou in Ref. [9], where they examine a mechanism by which small-scale fluctuations excite the large scale magnetic field. They modelled these fluctuations by an additive noise term in the mean field equation and identified the crucial role of non-normality on the dynamo process. Numerical simulations of the magnetoconvection equations were performed in Ref. [15] with analysis of effects of noise and non-normal transient growth. Inhomogeneous turbulent helicity fluctuations were considered in Ref. [16]. One should mention the dynamo model that exhibits aperiodic switching between regular behavior and chaotic oscillation [17]. Stochastic dynamo theory, using the term of an incoherent

dynamo, was proposed by Vishniac and Brandenburg in Ref. [18]. They showed how random fluctuations in the helicity can generate a large-scale magnetic field for the $\alpha\Omega$ -dynamo. We note that this model is closely related to the present work. However, they did not consider the transient amplification due to the non-normality of the dynamo equation. It turns out that non-normal dynamical systems exhibit an extraordinary sensitivity to random perturbations which leads to great amplification of the second moments of the stochastic dynamical systems (see Refs. [7,19,20]). Our recent work has demonstrated the possibility for stochastic magnetic energy amplification in the subcritical situation where the dynamo number is less than critical [10]. These observations motivate further studies of noise and nonlinearity effects which we consider below in the context of a simplified stochastic no- z model (see Ref. [21]). An issue we address in this paper is how the random fluctuations may be appropriately incorporated into the classical $\alpha\Omega$ -dynamo model. Our analysis especially focuses upon the multiplicative noise due to random fluctuations in the α -parameter and the non-normality of the dynamo equation operator. Recall that an operator is said to be non-normal if it does not commute with its adjoint in the corresponding scalar product. The determination of the effect of the noise, along with non-normality, on the amplification of magnetic energy during the early kinematic stage is the primary contribution of this paper.

In Section 2, we discuss the deterministic $\alpha\Omega$ -dynamo model, its equilibrium points and transient growth effects. In Section 3, we consider a linear stochastic model for the subcritical case. We derive equations for the second moments and find their stationary values. We demonstrate the important differences between the non-normal system and the normal one under the influence of additive noise. We explore the influence of multiplicative noise due to fluctuations in the α -parameter on the amplification of magnetic energy in the subcritical case during the kinematic stage. We derive the criteria under which the second moments grow exponentially with time (kinematic regime). Finally, in Section 4, we perform numerical experiments showing that there exists a series of noise-induced phase transitions in the $\alpha\Omega$ -dynamo model.

2. Stochastic $\alpha\Omega$ -dynamo model in a thin-disk approximation

Following [3,4] we consider here the thin-disk approximation to the dynamo equation for spiral galaxies, in which a turbulent disk of conducting fluid of uniform thickness $2h$ and radius R ($R \gg h$) rotates with angular velocity $\Omega(r)$. We restrict ourselves to the case of the $\alpha\Omega$ -dynamo for which the differential rotation dominates over the α -effect. The governing equations for the components of the axisymmetric magnetic field, $B_r(t, r, z)$ and $B_\phi(t, r, z)$ in the polar cylindrical coordinates (r, ϕ, z) can be written as,

$$\begin{aligned} \frac{\partial B_r}{\partial t} &= -\frac{\partial}{\partial z}(\alpha(t, z, r)B_\phi) + \beta \frac{\partial^2 B_r}{\partial z^2} + \frac{\beta}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial B_r}{\partial r} \right) + f_r(t, z, r), \\ \frac{\partial B_\phi}{\partial t} &= g_\omega B_r + \beta \frac{\partial^2 B_\phi}{\partial z^2} + \frac{\beta}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial B_r}{\partial r} \right) + f_\phi(t, z, r), \end{aligned} \quad (2)$$

where $g_\omega = r d\Omega/dr$ is the measure of differential rotation (usually $r d\Omega/dr < 0$), and $f_r(t, z, r)$ and $f_\varphi(t, z, r)$ are the stochastic terms describing unresolved turbulent fluctuations. The components $B_r(t, r, z)$ and $B_\varphi(t, r, z)$ obey vacuum boundary conditions on the thin disc surfaces $B_{r,\varphi}(t, r, -h) = 0$, $B_{r,\varphi}(t, r, h) = 0$ (see Refs. [3,4]).

In what follows we neglect the spatial structure of the magnetic field along the radius r and the height of the galaxy adopting the well-known no- z model [21]. The aim here is to concentrate on the studies of the influence of random fluctuations and non-normality on the dynamo process. We also take into account the nonlinear back-reaction [22,23]. The dynamical system for the azimuthal, $B_\varphi(t)$, and radial, $B_r(t)$, components of the magnetic field \mathbf{B} can be then written in the form

$$\begin{aligned} \frac{dB_r}{dt} &= -\frac{\alpha(|\mathbf{B}|)(1 + \xi_\alpha(t))}{h} B_\varphi - \frac{\pi^2 \beta(|\mathbf{B}|)}{4h^2} B_r + \xi_r(t), \\ \frac{dB_\varphi}{dt} &= g_\omega B_r - \frac{\pi^2 \beta(|\mathbf{B}|)}{4h^2} B_\varphi. \end{aligned} \quad (3)$$

This dynamical system may be regarded as a one-mode approximation of (2). It should be noted that the deterministic critical conditions for the generation of a magnetic field are the same for both models (2) and (3): $\alpha h^3 \beta^{-2} |g_\omega| > \pi^4/16$ [4]. Here we introduce the nonlinear functions $\alpha(|\mathbf{B}|)$ and $\beta(|\mathbf{B}|)$ describing the quenching of the α -effect and turbulent magnetic diffusivity. It is well-known that the current theories disagree about how the dynamo coefficients $\alpha(|\mathbf{B}|)$ and $\beta(|\mathbf{B}|)$ are suppressed by the mean field \mathbf{B} [22–26]. Here we describe the dynamo quenching by using the standard forms [5]

$$\alpha(|\mathbf{B}|) = \alpha_0 (1 + k_\alpha (B_\varphi/B_{eq})^2)^{-1}, \quad \beta(|\mathbf{B}|) = \beta_0 \left(1 + \frac{k_\beta}{1 + (B_{eq}/B_\varphi)^2} \right)^{-1}, \quad (4)$$

where k_α and k_β are constants of order one, and B_{eq} is the equipartition strength. We note that for the $\alpha\Omega$ -dynamo the azimuthal component $B_\varphi(t)$ is much larger than the radial field $B_r(t)$, therefore, $\mathbf{B}^2 \simeq B_\varphi^2$. The issue of the strong dependence of α and β on the magnetic Reynolds number R_m is still controversial and we did not include it in our analysis.

Regarding random perturbations in the system (3), we adopt a phenomenological mesoscopic approach in which the parameters of dynamo equations are assumed to be random functions of time [27,28]. The multiplicative noise $\xi_\alpha(t)$ in (3) stands for the rapid random fluctuations in the parameter α . One can show that they are more important than the random fluctuations in the turbulent magnetic diffusivity β [12–14]. The additive noise term $\xi_r(t)$ represents the stochastic forcing arising from the small-scale fluctuations of magnetic and turbulent velocity fields [9]. Both random terms are assumed to be independent Gaussian white noises with zero means $\langle \xi_\alpha(t) \rangle = 0$, $\langle \xi_r(t) \rangle = 0$ and correlations

$$\langle \xi_\alpha(t) \xi_\alpha(s) \rangle = 2D_\alpha \delta(t - s), \quad \langle \xi_r(t) \xi_r(s) \rangle = 2D_r \delta(t - s), \quad (5)$$

where D_α and D_r are the intensities of the noises and the angular brackets, $\langle \cdot \rangle$, denote the statistical average. One can show that the additive noise in the second equation in (3) is less important and can be ignored. In particular, the main contribution to the

second moment of B_φ comes from the additive noise in the first equation (see, for example, Ref. [20]). Of course, this paper addresses the over-simplified case of magnetic field generation in galaxies. Nonetheless, we present this work as an illustration of the influence the random fluctuations and non-normality may play in the dynamo process, and which therefore should be accounted for in more complicated dynamo modelling like the stochastic PDE (2).

The dynamical system (3) is well studied for the case when the noise terms are absent, and there has been much progress in the prediction of the growth rates induced by both the α -effect and differential rotation and the corresponding speed of magnetic waves. However, there is considerably less understanding of generation of a magnetic field in the presence of random fluctuations. Although the equations (3) are a theoretical simplification of what really happens in galaxies, we strongly believe that it does provide a useful framework for understanding the interaction of stochastic perturbation, non-normality and nonlinear effects.

It is convenient to rewrite the governing equations (3) in a nondimensional form by using an equipartition field strength B_{eq} , a length h , and a time Ω_0^{-1} , where Ω_0 is the typical value of angular velocity. In terms of the dimensionless parameters

$$g = \frac{|g_\omega|}{\Omega_0}, \quad \delta = \frac{R_\alpha}{R_\omega}, \quad \varepsilon = \frac{\pi^2}{4R_\omega}, \quad R_\alpha = \frac{\alpha_0 h}{\beta}, \quad R_\omega = \frac{\Omega_0 h^2}{\beta}, \quad (6)$$

the stochastic dynamo equations (3) can be written in the form of SDE [27,28]

$$dB_r = -(\delta\varphi_\alpha(B_\varphi)B_\varphi + \varepsilon\varphi_\beta(B_\varphi)B_r) dt - \sqrt{2\sigma_1}\varphi_\alpha(B_\varphi)B_\varphi dW_1 + \sqrt{2\sigma_2} dW_2, \quad (7)$$

$$dB_\varphi = -(gB_r + \varepsilon\varphi_\beta(B_\varphi)B_\varphi) dt, \quad (8)$$

where W_1 and W_2 are independent standard Wiener processes, and σ_1 and σ_2 are the noise intensities

$$\sigma_1 = \frac{D_\alpha}{h^2\Omega_0}, \quad \sigma_2 = \frac{D_r}{B_{eq}^2\Omega_0}. \quad (9)$$

Here we introduce the functions $\varphi_\alpha(B_\varphi)$ and $\varphi_\beta(B_\varphi)$ describing nonlinear quenching

$$\varphi_\alpha(B_\varphi) = \frac{1}{1 + k_\alpha B_\varphi^2}, \quad \varphi_\beta(B_\varphi) = \frac{1 + B_\varphi^2}{1 + (k_\beta + 1)B_\varphi^2}. \quad (10)$$

Now let us discuss the other parameters in Eqs. (7) and (8), namely, $\delta = R_\alpha/R_\omega$ and $\varepsilon = \pi^2/4R_\omega$. The parameter δ is the characteristic of relative importance of the ω -effect and the α -effect. For the $\alpha\Omega$ -dynamo the differential rotation dominates over the α -effect, that is, $R_\alpha \ll R_\omega$. Moreover, the diffusion time h^2/β is much larger than the time Ω_0^{-1} , therefore, both parameters δ and ε are small. Their typical values for spiral galaxies are $0.01 - 0.1$ ($R_\omega = 10 - 100$, $R_\alpha = 0.1 - 1$) [4]. It turns out that for small values of δ and ε , the linear operator (matrix) in Eqs. (7) and (8) is a highly non-normal one, since $g \sim 1$. This can lead to a large transient growth of the azimuthal component $B_\varphi(t)$ in the *subcritical* case. The nonlinear interactions may lead to a further amplification of this small disturbance [10]. The crucial idea behind subcritical transition is that the

α -effect or the ω -effect might be relatively weak, but the generation and maintenance of the large scale magnetic field is still possible.

3. Deterministic system

Let us briefly review the dynamics of the system (7) and (8) in the absence of noise terms. It takes the form

$$\begin{aligned} \frac{dB_r}{dt} &= -\delta\varphi_\alpha(B_\varphi)B_\varphi - \varepsilon\varphi_\beta(B_\varphi)B_r, \\ \frac{dB_\varphi}{dt} &= -gB_r - \varepsilon\varphi_\beta(B_\varphi)B_\varphi. \end{aligned} \tag{11}$$

The equilibrium points of the system (11) can be found from $\delta\varphi_\alpha(B_\varphi)B_\varphi + \varepsilon\varphi_\beta(B_\varphi)B_r = 0$, $gB_r + \varepsilon\varphi_\beta(B_\varphi)B_\varphi = 0$. The stationary value of the radial component, B_r , can be expressed in terms of the azimuthal one, B_φ ,

$$B_r = -\frac{\varepsilon\varphi_\beta(B_\varphi)B_\varphi}{g} \tag{12}$$

while B_φ is a solution of the equation

$$-\frac{\varepsilon^2}{g}\varphi_\beta^2(B_\varphi)B_\varphi + \delta\varphi_\alpha(B_\varphi)B_\varphi = 0. \tag{13}$$

Over a wide range of parameters, this equation might possess five solutions including the trivial one $B_\varphi = 0$. The other stationary points can be found from the equation

$$\frac{\varepsilon^2}{g\delta}(1 + B_\varphi^2)^2(1 + k_\alpha B_\varphi^2) - (1 + (k_\beta + 1)B_\varphi^2)^2 = 0. \tag{14}$$

Since there are multiple stable solutions (see Fig. 1 for $\varepsilon = 0.1$, $\delta = 0.01$, $k_\alpha = k_\beta = 1$), the system (11) can exhibit hysteresis [29].

Of course, the stochastic system, in which such multiple stable states can coexist (metastability), may exhibit random transitions between these states. In general, the transition rate is inversely proportional to the corresponding mean first passage time [27,28]. In the next sections we discuss the peculiarities of such transitions with non-normal effects.

The onset of the instability of the trivial equilibrium state $B_r = 0$, $B_\varphi = 0$ can be obtained from a standard linear stability analysis. Linearization gives

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B}, \tag{15}$$

where

$$\mathbf{A} = \begin{bmatrix} -\varepsilon & -\delta \\ -g & -\varepsilon \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_r \\ B_\varphi \end{bmatrix}. \tag{16}$$

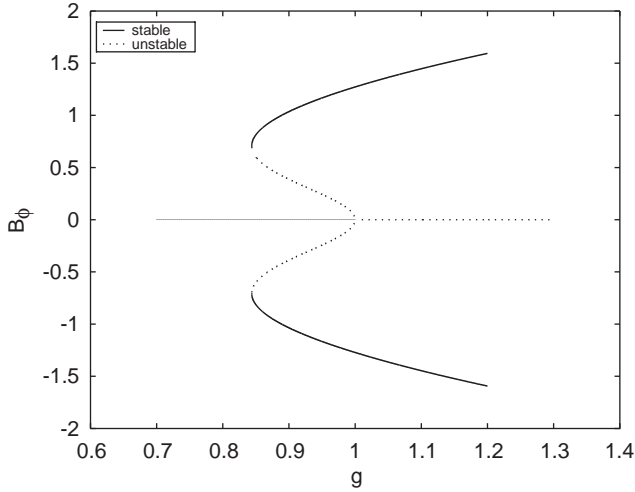


Fig. 1. The dependence of the stationary points of the deterministic system (11) on the parameter g (subcritical bifurcation).

The matrix \mathbf{A} has two eigenvalues

$$\lambda_1 = -\varepsilon + \sqrt{g\delta}, \quad \lambda_2 = -\varepsilon - \sqrt{g\delta}. \tag{17}$$

The *supercritical* instability condition ($\lambda_1 > 0$) can be written as $g\delta > \varepsilon^2$. The *subcritical* case corresponds to $g\delta < \varepsilon^2$. The main purpose of this paper is to study the stochastic amplification of the magnetic field \mathbf{B} in the *subcritical* case.

Since δ and ε are small parameters and $g \sim 1$, the matrix \mathbf{A} is a highly non-normal one. Recall that the matrix \mathbf{A} is normal, if $\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A}$, where T denotes the Hermitian transpose, otherwise it is non-normal. Even in the *subcritical case* when both eigenvalues $\lambda_{1,2}$ are negative, B_φ exhibits a large degree of transient growth before the exponential decay. The azimuthal component $B_\varphi(t)$, as a solution of the system (15) with the initial conditions

$$B_r(0) = -2c\sqrt{\delta/g}, \quad B_\varphi(0) = 0, \tag{18}$$

has the form

$$B_\varphi(t) = c(e^{\lambda_1 t} - e^{\lambda_2 t}). \tag{19}$$

Thus $B_\varphi(t)$ exhibits large transient growth over a timescale of order $1/(\lambda_2 - \lambda_1) \ln \lambda_1/\lambda_2$ before decaying exponentially. In other words the transient growth causes a temporary exit from the basin of attraction of the linearly stable solution $(0, 0)$. In Fig. 2 we plot phase portraits of the system (11) for $\varepsilon = 0.1$, $\delta = 0.01$, $k_x = k_\beta = 1$.

Similar deterministic low-dimensional models like (11) with transient growth have been proposed in Refs. [30–33] to explain the *subcritical* transition in the Navier-Stokes equations [34]. It will be interesting to analyze the so-called self-killing and self-creating dynamos by Fuchs, Rädler, and Rheinhard [35] in terms of nonlinear subcritical instability [10].

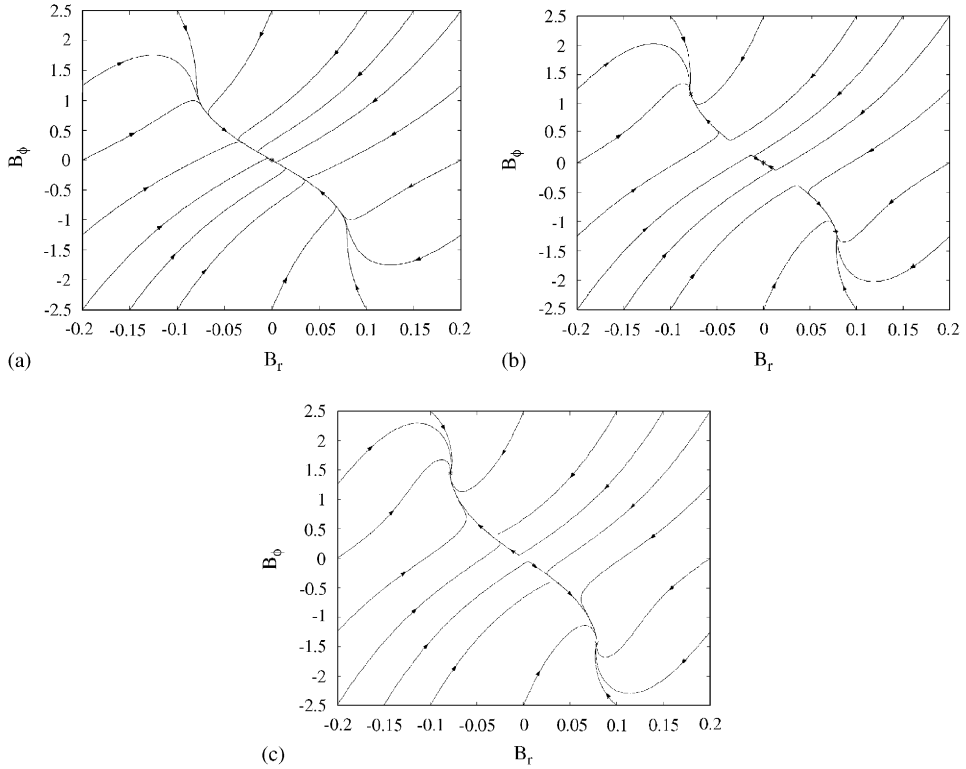


Fig. 2. Phase portraits of deterministic system (11): (a) $g=0.8$ one stable stationary point $(0, 0)$, (b) $g=0.95$ three stable stationary points, (c) $g=1.1$ two stable stationary points (supercritical case).

4. Analysis of linear stochastic system: subcritical case

In this section we discuss the circumstances under which the magnetic field of initially zero amplitude can experience sustained growth even in the subcritical case (kinematic regime). The linear approximation of the stochastic system (7), (8) near equilibrium point $B_r = 0, B_\phi = 0$ can be written as

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B} + \sqrt{2\sigma(\mathbf{B})}\mathbf{a} \frac{dW}{dt}, \tag{20}$$

$$\mathbf{A} = \begin{bmatrix} -\varepsilon & -\delta \\ -g & -\varepsilon \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_r \\ B_\phi \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{21}$$

$$\sigma(\mathbf{B}) = \sigma_1 B_\phi^2 + \sigma_2 = \sigma_1(\mathbf{B}, \mathbf{S}\mathbf{B}) + \sigma_2, \quad \mathbf{S} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{22}$$

Here W is the standard Wiener process.

The important statistical characteristics of this system are the second moments, since they represent the energy of the magnetic field (see, for example Refs. [9,10]). The second moments matrix $\mathbf{M} = E(\mathbf{B}\mathbf{B}^\top)$ is governed by system

$$\frac{d\mathbf{M}}{dt} = \mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{A}^\top + 2(\sigma_1 \text{tr}(\mathbf{M}\mathbf{S}) + \sigma_2)\mathbf{a}\mathbf{a}^\top. \tag{23}$$

There exists a variety of formal derivations of the Eq. (23). One of them can be found in Appendix A. It is also convenient to represent the matrix \mathbf{M} in the form

$$\mathbf{M} = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \tag{24}$$

where $m_1(t) = E(B_r^2)$, $m_2(t) = E(B_r B_\varphi)$, $m_3(t) = E(B_\varphi^2)$, and $E(\cdot)$ denotes the expectation.

4.1. Additive noise

In this section, we study how the additive noise can amplify the magnetic energy

$$E_B(t) \approx m_3(t) = E(B_\varphi^2) \tag{25}$$

during the kinematic stage. Without multiplicative noise ($\sigma_1 = 0$) the system (20) takes the form

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B} + \sqrt{2\sigma_2}\mathbf{a} \frac{dW}{dt}. \tag{26}$$

It follows from (23) that in the *subcritical* case the second moments matrix $\mathbf{M}(t)$ converges to a stationary value as $t \rightarrow \infty$ such that

$$m_1 = \frac{(2\varepsilon^2 - g\delta)\sigma_2}{2\varepsilon(\varepsilon^2 - g\delta)}, \quad m_2 = \frac{g\sigma_2}{2(g\delta - \varepsilon^2)}, \quad m_3 = \frac{g^2\sigma_2}{2\varepsilon(\varepsilon^2 - g\delta)}. \tag{27}$$

This result implies that additive noise effects, which occur naturally in the turbulent conducting fluids, together with differential rotation provide mechanisms which generate the large scale magnetic field in the subcritical case. Let us consider the dependence of the stationary value $m_3 = E(B_\varphi^2)$ on the differential rotation parameter g . When g varies from 0 to $g_{crit} = \varepsilon^2/\delta$ the function $m_3(g)$ increases such that $m_3(g) \rightarrow \infty$ as $g \rightarrow g_{crit} = \varepsilon^2/\delta$. Note that the sensitivity of normal dynamical systems to additive noise near the bifurcation point is a well-known result [27]. Let us discuss now the differences between a non-normal system like (26) and the normal one in the presence of additive noise. To understand this difference consider along with the non-normal system (26) the corresponding scalar normal stochastic equation

$$\frac{dB}{dt} = \lambda_1 B + \sqrt{2\sigma_2} \frac{dW}{dt}, \quad \lambda_1 = -\varepsilon + \sqrt{g\delta}. \tag{28}$$

In the *subcritical* case ($\lambda_1 < 0$), the solution of (28) is nothing else but the classical Ornstein-Uhlenbeck process [27,28]. It is easy to find the stationary value of its second moment $m = E(B^2)$:

$$m = \frac{\sigma_2}{\varepsilon - \sqrt{g\delta}}. \tag{29}$$

To assess the significance of the effect of non-normality with respect to the sensitivity to additive noise, let us introduce the new parameter $k = m_3/m$ that can be interpreted as a stochastic non-normality coefficient. If we consider the neighborhood of the bifurcation point $\lambda_1 = 0$, it gives us the measure of sensitivity of the non-normal system to noise compared to the normal one. By using (27) and (29), one can find

$$k = \frac{g^2}{2\varepsilon(\varepsilon + \sqrt{g\delta})}. \quad (30)$$

At the bifurcation point, $g = g_{crit}$ the value of this parameter is

$$k = \frac{g_{crit}^2}{4\varepsilon^2}. \quad (31)$$

Since $g_{crit} \sim 1$ it follows that $k \sim \varepsilon^{-2} \gg 1$, which shows how sensitive the non-normal system (26) is compared to the equivalent normal system (28). We note that the level of second moments maintained in stochastic non-normal dynamical systems associated with linearly stable shear flows has been discussed in Ref. [6].

4.2. Multiplicative noise: stochastic instability

In this section we discuss how the average magnetic energy $E_B(t) \approx m_3 = E(B_\varphi^2)$ is amplified by the random fluctuations of the α -parameter in the *subcritical* case (kinematic regime). In particular we find the critical value of the multiplicative noise intensity σ_{cr} such that for all values of $\sigma_1 > \sigma_{cr}$ the energy $E_B(t)$ grows as $\exp(\lambda t)$ during the early kinematic stage.

Consider the system (20) with multiplicative noise only

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B} + \sqrt{2\sigma_1(\mathbf{B}, \mathbf{S}\mathbf{B})}\mathbf{a} \frac{dW}{dt}. \quad (32)$$

It is well known that the second moments matrix of the system (20) converges to a stationary value as $t \rightarrow \infty$ if and only if the equilibrium $\mathbf{B} = 0$ of system (32) is exponentially stable in the mean square sense (EMS-stable). EMS-stability means that $E\|\mathbf{B}(t)\|^2 \leq K \exp(-lt)E\|\mathbf{B}(0)\|$, $K, l > 0$.

The equilibrium $\mathbf{B} = 0$ of system (32) is EMS-stable if and only if

- (a) the equilibrium $\mathbf{B} = 0$ of the deterministic system (11) is asymptotically stable;
- (b) $\text{tr}(\mathbf{M}\mathbf{S}) < 1$, where \mathbf{M} is the second moments stationary matrix for the system

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B} + \sqrt{2\sigma_1}\mathbf{a} \frac{dW}{dt}. \quad (33)$$

One can find the proof of this result in Appendix B. This criterion (see Ref. [36]) reduces the linear stability analysis of a system with multiplicative noise to that of the second moments stationary matrix of the corresponding system with the additive noise only.

For the *subcritical* case, it follows from this theorem that the system (20) is EMS-stable if

$$g^2 \sigma_1 < 2\varepsilon(\varepsilon^2 - g\delta). \quad (34)$$

The critical value of the multiplicative noise intensity σ_1 is

$$\sigma_{cr} = \frac{2\varepsilon(\varepsilon^2 - g\delta)}{g^2}. \quad (35)$$

If $\sigma_1 > \sigma_{cr}$ then for any additive noise intensity $\sigma_2 > 0$ the second moments of the system (32) tend to infinity as $t \rightarrow \infty$. This means that the random trajectories of the nonlinear system (7) and (8) leave the basin of attraction of zero equilibrium $B_r = 0$, $B_\varphi = 0$. The understanding of the stabilization of this growth should involve the numerical solution of the full nonlinear problem given by Eqs. (7) and (8).

It should be noted that the idea of a magnetic generation by differential rotation (the ω -effect) is widely accepted, but the α -effect is still considered as controversial. It follows from the above analysis that the amplification of magnetic energy might happen even in the case when the mean value of α is zero. It should be noted that this result is different from analogous in Ref. [16] since we consider here the second moments of the random magnetic field. Certainly, the predictions obtained here for the simplified stochastic model should be ultimately extended to more realistic systems of partial differential equations [37]. However, for now we use the no- z model as a simplified tool to observe the noise effects on the generation of magnetic energy in the *subcritical* case.

5. Numerical analysis: nonlinear case

So far we have concentrated on the linear stochastic instability where the second moments grow exponentially without limit (kinematic regime). Obviously, in the nonlinear case, if we take into account the backreaction which suppresses the effective dissipation β , and the α -effect, one can expect an entirely different global behavior.

In this section, we perform simulations of random trajectories of the nonlinear dynamical system (7), (8) for $\varepsilon = 0.1$, $\delta = 0.01$, $k_\alpha = k_\beta = 1$. In this case the critical value of the differential rotation parameter g_{crit} is 1. Since we are interested in the *subcritical* case, $g < 1$, we choose g to be 0.99. It follows from (35) that for this set of parameters the critical value of multiplicative noise intensity $\sigma_{cr} = 2 \times 10^{-5}$. Let us emphasize again that the main reason why σ_{cr} is so small is because of the high level of non-normality of the dynamical system (7) and (8).

Fig. 3 demonstrates qualitative changes in the shapes of the probability density function (pdf) of B_φ for fixed t as the intensity of multiplicative noise σ_1 increases ($\sigma_2 = 0.5 \times 10^{-7}$).

Here asterisks mark the positions of the equilibrium points for the deterministic system. One can see that there are three qualitatively different regimes depending on the value σ_1 . For very small values of σ_1 the pdf is concentrated around the equilibrium point $B_\varphi = 0$. Stochastic trajectories of (7) and (8) calculated by direct numerical simulations are shown in Figs. 4 and 5.

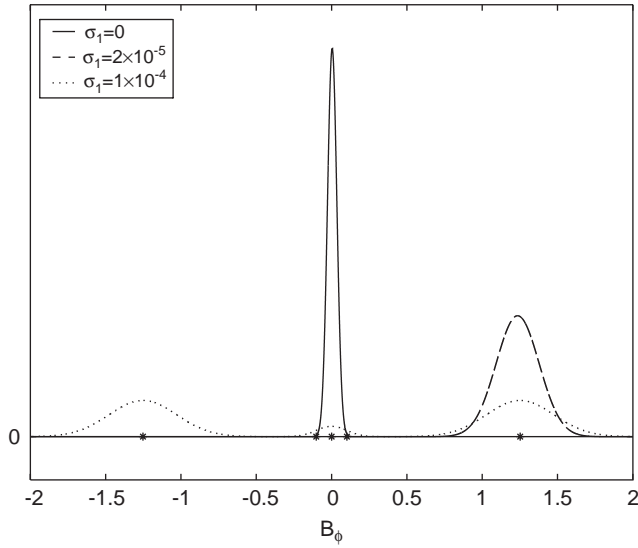


Fig. 3. Probability density functions for different values of the intensity of multiplicative noise.

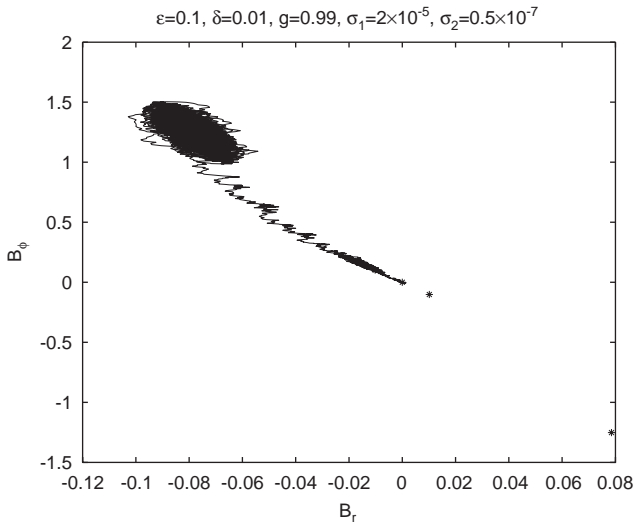


Fig. 4. Stochastic trajectory escapes from zero equilibrium point and concentrates near non-zero equilibrium.

One can see that as σ_1 increases, the trajectories (see Fig. 4) escape from the domain of attraction of the zero equilibrium point and concentrate in the vicinity of the non-zero stable equilibrium point. Further increase of σ_1 leads to quite complicated dynamics

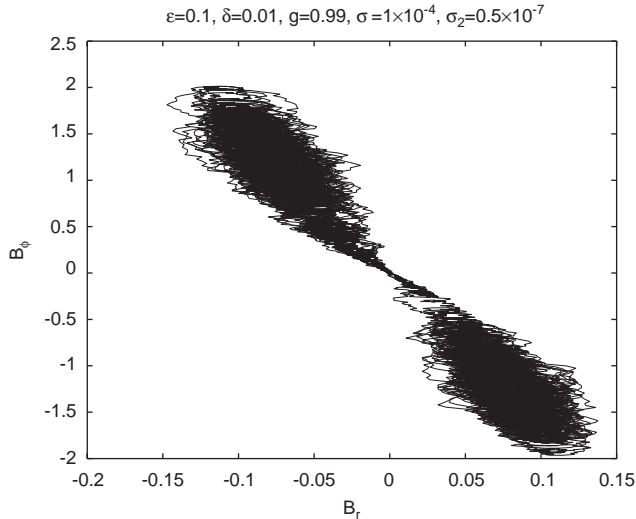


Fig. 5. Stochastic trajectory moves around all stationary points.

when the stochastic trajectory (see Fig. 5) moves around all deterministic stationary points.

6. Discussion and conclusions

We have studied the stochastic amplification of large scale magnetic fields in a differentially rotating system in a subcritical regime and discussed the possible implications of this for the magnetic field in galaxies. The main purpose was to address the stochastic generation that cannot be explained by traditional linear eigenvalue analysis. We have chosen the simplified stochastic $\alpha\Omega$ -dynamo model for galaxies in a thin-disk approximation and thereby concentrated on the influence of additive and multiplicative noises along with non-normality on the amplification of the magnetic field in the subcritical case (when the dynamo number is less than critical). In the linear case, we have derived the equations for second moments describing the magnetic energy and demonstrated the important differences between the non-normal system and the normal one under the influence of additive noise. For the multiplicative noise, the criteria for the stochastic instability during the early kinematic stage was established in terms of the critical value of noise intensity due to α -fluctuations. In the non-linear case, we have performed numerical simulations of nonlinear stochastic differential equations for the $\alpha\Omega$ -dynamo and found a series of noise induced phase transitions: qualitative changes in the behavior of the trajectories due to the increase in the noise intensity parameter.

It should be noted that Eqs. (7) and (8) as applied to galaxies are a theoretical simplification, but they do provide a useful framework for understanding the effect of

random fluctuations. Our finding for the stochastic parametric instability can be straightforwardly applied to partial differential equation (2) in which case one has to consider the coupled system for eigenmodes [12]. In this respect the two Eqs. (7) and (8) can be regarded as a dynamical system for first eigenmodes (order parameters), where the influence of other degrees of freedom is parameterized by additive and multiplicative noises. We believe that our theory can also be extended and applied (after some modifications) to the solar dynamo in which a toroidal field is generated by the action of a shear flow (typical non-normal effect).

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Appendix A

A.1. Formal derivation of the equation for the second moments matrix

Let $\Delta\mathbf{B}=\mathbf{B}(t+\Delta t)-\mathbf{B}(t)$. It follows formally from Eq. (20) that $\Delta\mathbf{B}=\mathbf{A}\mathbf{B}\Delta t+\mathbf{v}\Delta W$, where $\mathbf{B}=\mathbf{B}(t)$, $\mathbf{v}=\sqrt{2\sigma(\mathbf{B}(t))}\cdot\mathbf{a}$ or

$$\mathbf{B}(t+\Delta t)=\mathbf{B}+\mathbf{A}\mathbf{B}\Delta t+\mathbf{v}\Delta W. \quad (\text{A.1})$$

Eq. (A.1) implies

$$\begin{aligned} \mathbf{B}(t+\Delta t)\mathbf{B}^\top(t+\Delta t) &= (\mathbf{B}+\mathbf{A}\mathbf{B}\Delta t+\mathbf{v}\Delta W)(\mathbf{B}^\top+\mathbf{B}^\top\mathbf{A}^\top\Delta t+\mathbf{v}^\top\Delta W) \\ &= \mathbf{B}\mathbf{B}^\top+(\mathbf{A}\mathbf{B}\mathbf{B}^\top+\mathbf{B}\mathbf{B}^\top\mathbf{A}^\top)\Delta t+\mathbf{v}\mathbf{v}^\top(\Delta W)^2 \\ &\quad +(\mathbf{v}\mathbf{B}^\top+\mathbf{B}\mathbf{v}^\top)\Delta W+(\mathbf{A}\mathbf{B}\mathbf{v}^\top+\mathbf{v}\mathbf{B}^\top\mathbf{A}^\top)\Delta t\Delta W \\ &\quad +\mathbf{A}\mathbf{B}\mathbf{B}^\top\mathbf{A}^\top(\Delta t)^2. \end{aligned} \quad (\text{A.2})$$

Taking into account the standard relations $E\Delta W=0$, $E(\Delta W)^2=\Delta t$ for a Wiener process and $E(\mathbf{v}\mathbf{v}^\top)=2(\sigma_1\text{tr}(\mathbf{M}\mathbf{S})+\sigma_2)\mathbf{a}\mathbf{a}^\top$ we can find from (A.2) for $\mathbf{M}(t)=E(\mathbf{B}(t)\mathbf{B}^\top(t))$ the following equation

$$\begin{aligned} \Delta\mathbf{M} &= \mathbf{M}(t+\Delta t)-\mathbf{M}(t) \\ &= (\mathbf{A}\mathbf{M}+\mathbf{M}\mathbf{A}^\top+2(\sigma_1\text{tr}(\mathbf{M}\mathbf{S})+\sigma_2)\mathbf{a}\mathbf{a}^\top)\Delta t+\mathbf{A}\mathbf{M}\mathbf{A}^\top(\Delta t)^2. \end{aligned} \quad (\text{A.3})$$

Now system (23) follows from (A.3) immediately.

Appendix B

Necessity. From EMS-stability of (32) it follows that there exists a stationary second moments matrix \mathbf{M} of system (20) satisfying the equation

$$\mathbf{A}\mathbf{M} + \mathbf{M}\mathbf{A}^\top + 2(\sigma_1 \text{tr}(\mathbf{M}\mathbf{S}) + \sigma_2)\mathbf{a}\mathbf{a}^\top = 0. \tag{B.1}$$

One can find for $\sigma_1 > 0$ that the matrix

$$\bar{\mathbf{M}} = \mathbf{M} / \left(\text{tr}(\mathbf{M}\mathbf{S}) + \frac{\sigma_2}{\sigma_1} \right) \tag{B.2}$$

is a solution of the equation

$$\mathbf{A}\bar{\mathbf{M}} + \bar{\mathbf{M}}\mathbf{A}^\top + 2\sigma_1\mathbf{a}\mathbf{a}^\top = 0. \tag{B.3}$$

Note that $\bar{\mathbf{M}}$ is a stationary second moments matrix for the system (33). The obvious inequality

$$\text{tr}(\bar{\mathbf{M}}\mathbf{S}) = \frac{\text{tr}(\mathbf{M}\mathbf{S})}{\text{tr}(\mathbf{M}\mathbf{S}) + \frac{\sigma_2}{\sigma_1}} < 1 \tag{B.4}$$

proves necessity.

Sufficiency. Consider a stationary second moments matrix \mathbf{M} of system (33) satisfying Eq. (B.3) and the inequality $\text{tr}(\mathbf{M}\mathbf{S}) < 1$. It means for sufficiently small $\varepsilon > 0$ a solution \mathbf{M}_ε of the equation

$$\mathbf{A}\mathbf{M}_\varepsilon + \mathbf{M}_\varepsilon\mathbf{A}^\top + 2\sigma_1\mathbf{a}\mathbf{a}^\top + 2\varepsilon\mathbf{I} = 0, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.5}$$

satisfies the inequality

$$\text{tr}(\mathbf{M}_\varepsilon\mathbf{S}) < 1 \tag{B.6}$$

too. From (B.5) and (B.6) it follows that

$$\mathbf{A}\mathbf{M}_\varepsilon + \mathbf{M}_\varepsilon\mathbf{A}^\top + 2(\sigma_1 \text{tr}(\mathbf{M}_\varepsilon\mathbf{S}) + \bar{\sigma}_2)\mathbf{a}\mathbf{a}^\top + 2\varepsilon\mathbf{I} = 0, \tag{B.7}$$

where $\bar{\sigma}_2 = \sigma_1(1 - \text{tr}(\mathbf{M}_\varepsilon\mathbf{S})) > 0$. Formula (B.7) shows that \mathbf{M}_ε is a stationary second moments matrix for the stochastic system

$$\frac{d\mathbf{B}}{dt} = \mathbf{A}\mathbf{B} + \sqrt{2(\sigma_1(\mathbf{B}, \mathbf{S}\mathbf{B}) + \bar{\sigma}_2)}\mathbf{a} \frac{dW}{dt} + \sqrt{2\varepsilon} \frac{d\xi}{dt}, \tag{B.8}$$

where ξ is the independent two-dimensional standard Wiener process. The existence of a stationary second moments matrix for the stochastic system (B.8) with two-dimensional nondegenerate additive noise of intensity $\varepsilon > 0$ proves EMS-stability of the system (32). The details of the proof can be found in Ref. [36].

References

[1] H.K. Moffatt, Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Press, New York, 1978.

- [2] F. Krause, K.-H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory*, Academic-Verlag, Berlin, 1980.
- [3] Ya.B. Zeldovich, A.A. Ruzmaikin, D.D. Sokoloff, *Magnetic Fields in Astrophysics*, Gordon and Breach Science Publishers, New York, 1983.
- [4] A.A. Ruzmaikin, A.M. Shukurov, D.D. Sokoloff, *Magnetic Fields in Galaxies*, Kluwer Academic Publishers, Dordrecht, 1988.
- [5] L. Widrow, *Rev. Mod. Phys.* 74 (2002) 775.
- [6] B.F. Farrell, P.J. Ioannou, *Phys. Fluids* 5 (1993) 2600.
- [7] B.F. Farrell, P.J. Ioannou, *Phys. Rev. Lett.* 72 (1994) 1188.
- [8] B.F. Farrell, P.J. Ioannou, *J. Atmos. Sci.* 53 (1996) 2025.
- [9] B. Farrel, P. Ioannou, *Astrophys. J.* 522 (1999) 1088.
- [10] S. Fedotov, *Phys. Rev. E* 68 (2003) 067301.
- [11] R. Kraichnan, *J. Fluid Mech.* 75 (1976) 657.
- [12] P. Hoyng, *Astron. Astrophys.* 171 (1987) 348.
- [13] P. Hoyng, D. Schmitt, L.J.W. Teuben, *Astron. Astrophys.* 289 (1994) 265.
- [14] P. Hoyng, N.A.J. Schutgens, *Astron. Astrophys.* 293 (1995) 777.
- [15] J.R. Gog, I. Opera, M.R.E. Proctor, A.M. Rucklidge, *Proc. R. Soc. Lond. A* 455 (1999) 4205.
- [16] N.A. Silant'ev, *Astron. Astrophys.* 364 (2000) 339.
- [17] N. Platt, E.A. Spiegel, C. Tresser, *Geophys. Astrophys. Fluid Dyn.* 73 (1993) 146.
- [18] E. Vishniac, A. Brandenburg, *Astrophys. J.* 475 (1997) 263.
- [19] S. Grossmann, *Rev. Mod. Phys.* 72 (2000) 603.
- [20] S. Fedotov, I. Bashkirtseva, L. Ryashko, *Phys. Rev. E* 66 (2002) 066310.
- [21] D. Moss, *Mon. Not. R. Astrophys. Soc.* 275 (1995) 191.
- [22] F. Cattaneo, D.W. Hughes, *Phys. Rev. E* 54 (1996) R4532.
- [23] A.V. Gruzinov, P.H. Diamond, *Phys. Rev. Lett.* 72 (1994) 1651.
- [24] A. Brandenburg, *Astrophys. J.* 550 (2001) 824.
- [25] I. Rogachevskii, N. Kleeorin, *Phys. Rev. E* 64 (2001) 056307.
- [26] E.G. Blackman, G.B. Field, *Phys. Rev. Lett.* 89 (2002) 265007.
- [27] C.W. Gardiner, *Handbook of Stochastic Methods*, 2nd Edition, Springer, New York, 1996.
- [28] W. Horsthemke, R. Lefever, *Noise-Induced Transitions*, Springer, Berlin, 1984.
- [29] S.M. Tobias, *Astrophys. J.* 467 (1996) 870.
- [30] L.N. Trefethen, A.E. Trefethen, S.C. Reddy, T.A. Driscoll, *Science* 261 (1993) 578.
- [31] T. Gebhardt, S. Grossmann, *Phys. Rev. E* 50 (1994) 3705.
- [32] J.S. Baggett, T.A. Driscoll, L.N. Trefethen, *Phys. Fluids* 7 (1995) 833.
- [33] J.S. Baggett, L.N. Trefethen, *Phys. Fluids* 9 (1997) 1043.
- [34] P.J. Schmid, D.S. Henningson, *Stability and Transition in Shear Flows*, Springer, Berlin, 2001.
- [35] H. Fuchs, K.-H. Rädler, M. Rheinhard, *Astron. Nachr.* 320 (1999) 127.
- [36] L.B. Ryashko, *Prikl. Mat. Mech.* 45 (1981) 778.
- [37] R. Beck, A. Brandenburg, D. Moss, A. Shukurov, D. Sokoloff, *Ann. Rev. Astron. Astrophys.* 34 (1996) 155.