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# OPTION PRICING FOR INCOMPLETE MARKETS VIA STOCHASTIC OPTIMIZATION: TRANSACTION COSTS, ADAPTIVE CONTROL AND FORECAST

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The problem of determining the European-style option price in incomplete markets is examined within the framework of stochastic optimization. An analytic method based on the stochastic optimization is developed that gives the general formalism for determining the option price and the optimal trading strategy (optimal feedback control) that reduces the total risk inherent in writing the option. The cases involving transaction costs, the stochastic volatility with uncertainty, stochastic adaptive process, and forecasting process are considered. A software package for the option pricing for incomplete markets is developed and the results of numerical simulations are presented.

*Keywords*: Option pricing, incomplete markets, stochastic optimization, transaction costs, adaptive control, forecast.

## 1. Introduction

An essential feature of the currently dominant option pricing theory proposed by Black and Scholes is the existence of a dynamic trading strategy in the underlying asset that exactly replicates the derivative contract payoff [1–4]. However, in general, the market is not complete, the contingent claim is not a redundant asset and therefore its price cannot be determined by the no-arbitrage argument alone. The reasons that give rise to an incompleteness of market might be very different, for example, mixed jump-diffusion price process for an asset [1, 5], stochastic volatility [6], etc.

In recent years there has been a substantial theoretical effort to give the pricing formula for a derivative security for which an exact replicating portfolio in the underlying asset ceases to exist. The typical example involving incompleteness is a

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model in which the stock volatility is a stochastic process. Several approaches to the valuation of the contingent claim under random volatility have been suggested in the literature [6–11]. Typically the pricing formulas involve the unobservable parameter, so-called market price of volatility risk. This fundamental difficulty has led researches to accept the idea of uncertain volatility when all prices for contingent claim are possible within some specified range [12–15].

An alternative method for the derivative pricing in the incomplete markets has been proposed in a series of papers by mathematicians Müller, Fölmer, Sondermann, Schweizer and Schäl [16–20] and by physicists Bouchaud and Sornette [21] (see also [22–25]). The basic idea is that the fair price of a contingent claim can be found through a risk minimization procedure. Different criteria for measuring the risk inherent in writing an option have been suggested, including the global and local variance of the cost process [16–18], and the variance of the global operator wealth [19, 21–23].

Although significant progress has already been made in the option pricing theory involving the risk minimization procedure, there still exist many open problems including how to derive an *effective algorithm giving the option price and trading strategy* involving stochastic volatility with uncertainty, adaptive decision process, forecast, transaction costs, etc. The basic purpose of this paper is to present such an algorithm that can be used in practice. In particular we intend to show: (i) how the problem of option pricing based on the risk minimization analysis can be reformulated in terms of Maier's problem and (ii) how the stochastic optimization procedure [26] based on the Bellman equation can be implemented to give a reliable numerical technique for determining both the derivative price and the optimal trading strategy. It should be noted that the application of dynamic programming approach to option pricing can be found in [18, 28, 29, 35].

## 2. Statement of the Problem

We consider a model of $(S, B)$ market handling at discrete times $n = 0, 1, \ldots, N$. The market consists of two assets: a stock $S_n$, risky asset, whose price dynamics is governed by the stochastic difference equation

$$S_{n+1} = (1 + \xi_n)S_n\,, \quad S_0 > 0\,, \tag{1}$$

where $\xi_n$ is a sequence of independent random variables, and a bond, $B_n$, the riskless asset, whose price dynamics is given by recurrent relation

$$B_{n+1} = (1 + r)B_n\,, \quad B_0 > 0 \tag{2}$$

with interest rate $r > 0$.

We assume that at time zero an investor sells an European-style option with the strike price $X$ for $C_0$ and invests this money in a portfolio containing $\Delta_0$ shares and $\theta_0$ bonds. The initial value $V_0$ of this portfolio is given by

$$V_0 = C_0 = \Delta_0 S_0 + \theta_0 B_0\,. \tag{3}$$

The investor is interested in constructing the self-financing strategy to hedge the option exposure. Since for incomplete markets the exact replication of the option payoffs by a portfolio of traded securities is not possible, the investor cannot completely neutralize the risk inherent in writing the option. Hence the problem is to find such a trading strategy that reduces total risk to some intrinsic value [16–25].

To proceed further we need an equation governing the dynamics of the self-financing hedged portfolio. First we consider the case of frictionless trading. The case involving the transaction costs will be considered in Sec. 4. The value of the portfolio $V_n$ at time $n$ may be written as

$$V_n = \Delta_n S_n + \theta_n B_n\,, \tag{4}$$

where $\Delta_n$ is the number of shares of the underlying asset and $n$ the number of bonds during the time interval $[n, n+1)$. By using (4) we can find the variation of portfolio at two successive moments of time in the form,

$$V_{n+1} - V_n = \Delta_n(S_{n+1} - S_n) + \theta_n(B_{n+1} - B_n)\,, \tag{5}$$

where we use self-financed trading strategy condition

$$(\Delta_{n+1} - \Delta_n)S_{n+1} + (\theta_{n+1} - \theta_n)B_{n+1} = 0. \tag{6}$$

The intuitive interpretation of the last equation is simple: movement of the capital in a bank account $(\theta_{n+1} - \theta_n)B_{n+1}$ can occur only due to identical alteration of capital in stock $(\Delta_{n+1} - \Delta_n)S_{n+1}$. Substitution of $S_{n+1}$ and $B_{n+1}$ in (5) from (1) and (2) gives

$$V_{n+1} = (1 + r)V_n + \Delta_n(\xi_n - r)S_n\,. \tag{7}$$

Following [19] we propose that the investor's purpose is to maintain a self-financed portfolio (4) in such a way that at the expiration date $N$ the terminal value of this portfolio

$$V_N = (1 + r)V_{N-1} + \Delta_{N-1}(\xi_{N-1} - r)S_{N-1} \tag{8}$$

should be as close as possible to the option payoff

$$P_X(S_N) \equiv \max(S_N - X, 0)\,. \tag{9}$$

One way to achieve this purpose is to require that the expectation value of the difference between the option value and the value of hedged portfolio at expiration is equal to zero

$$E\{P_X(S_N) - V_N\} = 0 \tag{10}$$

while the variance of this difference

$$R = E\{(P_X(S_N) - V_N)^2\} \tag{11}$$

as a measure of the risk should be minimized by the proper choice of the trading strategy $\{\Delta_0, \ldots, \Delta_{N-1}\}$. The operator $E\{\cdot\}$ denotes expectation with respect to the distributions of $\xi_0, \xi_1, \ldots, \xi_{N-1}$.

It follows from (7) that the portfolio at the terminal date $N$ can be written in the form

$$V_N = (1+r)^N C_0 + \sum_{n=0}^{N-1} (1+r)^{N-n-1} \Delta_n (\xi - r) S_n \,. \tag{12}$$

The initial investment $C_0$ required to fund the dynamical hedged portfolio is nothing else but the price of the option which should also minimize the risk $R$ in (11), that is

$$\partial R / \partial C_0 = 0 \,. \tag{13}$$

By using (11) and (12) we can find that

$$\partial R / \partial C_0 = 2(1+r)^N E\{P_X(S_N) - V_N\} \,. \tag{14}$$

It follows from (14) that the requirements (10) and (13) are equivalent and therefore we only consider here the risk minimization problem

$$\min_{\Delta_1 \cdots \Delta_{N-1}, C_0} E\{(P_X(S_N) - V_N)^2\} \,. \tag{15}$$

## 3. Stochastic Optimization

According to the ideas of dynamic programming [26, 27], the proper choice of the sequence controls $\Delta_n$ should involve the information aggregation, i.e. the optimal choice of trading strategy at each of $N$ time periods should be based on available information about the current values of asset price and hedged portfolio. From a mathematical point of view it means that one has to find a sequence of functions (so-called optimal control policy)

$$\Delta_n^* = \Delta_n^*(S_n, V_n) \tag{16}$$

that minimizes the total risk. In what follows we will use (4) to find an optimal value of $\theta_n$, that is

$$\theta_n^*(S_n, V_n) = (V_n - \Delta_n^*(S_n, V_n) S_n) B_n^{-1} \,. \tag{17}$$

Let us consider the problem of minimizing the risk-function

$$R = E\{(P_X(S_N) - V_N)^2\} \,.$$

Following the dynamic programming approach, we first consider the last time period and proceed backwards in time. If at the beginning of the last trading period $N-1$ the stock price is $S_{N-1}$ and the value of portfolio is $V_{N-1}$, then no matter what happened in the past periods, the investor should choose such a trading strategy $\Delta_{N-1}$, $\theta_{N-1}$ that minimizes the risk for the last time period.

Let us introduce the minimal risk for the last period, which is a function of the stock price $S_{N-1}$ and the value of the portfolio $V_{N-1}$

$$R_{N-1}(S_{N-1}, V_{N-1}) = \min_{\Delta_{N-1}} E_{\xi_{N-1}}\{(P_X(S_N) - V_N)^2\} \,. \tag{18}$$

It follows from (1) and (8) that $R_{N-1}$ can be rewritten as

$$R_{N-1}(S_{N-1}, V_{N-1}) = \min_{\Delta_{N-1}} E_{\xi_{N-1}}\{(P_X(S_{N-1} + \xi_{N-1}S_{N-1}) - (1+r)V_{N-1} - \Delta_{N-1}(\xi_{N-1} - r)S_{N-1})^2\}\,. \tag{19}$$

By calculating this function we obtain the optimal value of $\Delta_{N-1}$ and thereby the optimal trading policy $\Delta^*_{N-1}(S_{N-1}, V_{N-1}), \theta^*_{N-1}(S_{N-1}, V_{N-1})$ for the last period.

At the beginning of time period $N-2$ when the stock price is $S_{N-2}$ and the value of the portfolio is $V_{N-2}$, the investor should readjust the position in a such way that $\Delta_{N-2}$ minimizes the risk $E_{\xi_{N-2}}\{R_{N-1}(S_{N-1}, V_{N-1})\}$. The dynamic programming algorithm thus takes the form of the recurrence equation

$$R_{N-2}(S_{N-2}, V_{N-2}) = \min_{\Delta_{N-2}} E_{\xi_{N-2}}\{R_{N-1}(S_{N-2} + \xi_{N-2}S_{N-2}, (1+r)V_{N-2} + \Delta_{N-2}(\xi_{N-2} - r)S_{N-2})\}\,. \tag{20}$$

By calculating $R_{N-2}(S_{N-2}, V_{N-2})$ we obtain the optimal function $\Delta^*_{N-2}(S_{N-2}, V_{N-2})$.

Repeating these arguments we can get the Bellman equation for the period $n$

$$R_n(S_n, V_n) = \min_{\Delta_n} E_{\xi_n}\{R_{n+1}(S_n + \xi_n S_n, (1+r)V_n + \Delta_n(\xi_n - r)S_n)\}\,. \tag{21}$$

At the last stage the option price $C_0$, a number of stocks in optimal portfolio $\Delta_0$ and the residual risk $R_0$ can be obtained from minimum condition

$$R_0(S_0) = \min_{\Delta_0, C_0} E_{\xi_0}\{R_1(S_0 + \xi_0 S_0, (1+r)V_0 + \Delta_0(\xi_0(\xi_0 - r)S_0)\}\,. \tag{22}$$

The attractive feature of the dynamic programming algorithm is the relative simplicity with which the optimal trading policy $\Delta^*_n(S_n, V_n), \theta^*_n(S_n, V_n)$ can be computed. The basic advantage of the general algorithm (21) over the functional derivative technique [21] is that the original problem (15) is reduced to a sequence of minimization problems, each of which is much simpler than the original one.

It might seem that the better choice of control in (21) would be a pair $(\Delta_n, \theta_n)$ giving the control policy $\Delta^*_n(S_n), \theta^*_n(S_n)$ as functions of the asset price $S_n$ only. However the self-financing condition gives rise to the restriction (5) that makes the control problem in terms of the pair $(\Delta_n, \theta_n)$ rather difficult.

## 4. Transaction Costs

Let us now consider the problem of finding the optimal trading strategy and the option price in the presence of transaction costs. We know that the effects of transaction costs on the contingent claim pricing might be very complex depending on the size of bid-offer spreads, the structure of payoff functions, etc. [4, 30–35]. Here we suggest a new algorithm for a valuation of option price based on the risk minimization procedure.

We assume a bid-offer spread in which the investor buys the stock for the offer price $S(1+k)$ and sells it for the bid price $S(1-k)$ admitting a loss of $2kS$ in cash. Again we formulate the problem in terms of an investor who sells the European option with payoff $P_X(S_N)$ and who employs the trading strategy to hedge the derivative. At time zero a hedged portfolio is constructed by purchasing $\Delta_0$ shares at the offer price $S_0(1+k)$ and $\theta_0$ bonds so that the amount of money spent for this portfolio including the effect of transaction cost can be written as

$$V_0 = \Delta_0 S_0 + \theta_0 B_0 + k\Delta_0 S_0 \,.$$

It is assumed here that the investor has no initial position in the underlying asset. The investor's purpose is to maintain a dynamic portfolio strategy in such way that the risk of his liability (11) is minimal. To proceed further we need an equation governing the dynamics of the self-financing hedged portfolio. We consider the trading with transactions costs. The transaction costs of trading $\Delta_{n+1} - \Delta_n$ shares is $k|\Delta_{n+1} - \Delta_n|S_{n+1}$ therefore the value of portfolio $V_n$ at time $n$ may be written as

$$V_n = \Delta_n S_n + \theta_n B_n + k|\Delta_n - \Delta_{n-1}|S_n \,. \tag{23}$$

In contrast to frictionless trading (3)–(8) each trading gives a loss of $k|\Delta_{n+1} - \Delta_n|S_{n+1}$ in cash. Therefore in this case the self-financing requirement (6) has the following form

$$(\Delta_{n+1} - \Delta_n)S_{n+1} + (\theta_{n+1} - \theta_n)B_{n+1} = -k|\Delta_{n+1} - \Delta_n|S_{n+1} \,. \tag{24}$$

It follows from (1), (2), (23) and (24) that the difference equation for portfolio dynamics can be written as

$$V_{n+1} = (1+r)V_n + \Delta_n(\xi_n - r)S_n - (1+r)k|\Delta_n - \Delta_{n-1}|S_n \,. \tag{25}$$

Above we have derived a Bellman equation (21) when the asset price and the value of portfolio have been chosen as the dynamical variables, while the number of shares in the portfolio has played the role of the control parameter. In the presence of transaction costs it is necessary to introduce a new state variable $\Omega_n$ such that

$$\Omega_{n+1} = \Delta_n \,, \tag{26}$$

so that the dynamical variable $V_n$ obeys the stochastic difference equation

$$\begin{aligned} V_{n+1} &= (1+r)V_n + \Delta_n(\xi_n - r)S_n - (1+r)k|\Delta_n - \Omega_n|S_n \,, \\ V_1 &= (1+r)V_0 + \Delta_0(\xi_0 - r)S_0 - (1+r)k\Delta_0 S_0 \,. \end{aligned} \tag{27}$$

Now we are in a position to formulate the basic problem. If we introduce the minimal risk for the last period which is a function of $S_{N-1}, V_{N-1}$ and $\Omega_{N-1}$

$$R_{N-1}(S_{N-1}, V_{N-1}, \Omega_{N-1}) = \min_{\Delta_{N-1}} E_{\xi_{N-1}}\{(P_X(S_N) - V_N)^2\} \,,$$

it follows from (1) and (27) that

$$R_{N-1}(S_{N-1}, V_{N-1}, \Omega_{N-1}) = \min_{\Delta_{N-1}} E_{\xi_{N-1}}\{(P_X(S_{N-1} + \xi_{N-1}S_{N-1}) - (1+r)V_{N-1} - \Delta_{N-1}(\xi_{N-1} - r)S_{N-1} + (1+r)k|\Delta_{N-1} - \Omega_{N-1}|S_{N-1})^2\}\,. \tag{28}$$

By a minimization procedure we obtain the optimal value of $\Delta_{N-1}$ and thereby the optimal trading policy $\Delta^*_{N-1}(S_{N-1}, V_{N-1}, \Omega_{N-1})$ for the last period. It is clear from (26) that the optimal control policy $\Delta^*_{N-1}$ is a function of the present state $(S_{N-1}, V_{N-1})$ as well as past control $\Delta_{N-2}$.

The principle of optimality yields the general recurrence relation

$$R_n(S_n, V_n, \Omega_n) = \min_{\Delta_n} E_{\xi_n}\{R_{n+1}(S_n + \xi_n S_n, (1+r)V_n + \Delta_n(\xi_n - r)S_n - (1+r)k|\Delta_n - \Omega_n|S_n, \Delta_n)\}\,. \tag{29}$$

Let us denote by $R_0(S_0, V_0, )$ the minimal risk that can be achieved by starting from the arbitrary initial state $S_0, V_0$

$$R_0(S_0, V_0) = \min_{\Delta_0 \cdots \Delta_{N-1}} E\{(P_X(S_n) - V_N)^2\}\,,$$

then

$$R_0(S_0, V_0) = \min_{\Delta_0} E_{\xi_0}\{R_1(S_0 + \xi_0 S_0, (1+r)V_0 + \Delta_0(\xi_0 - r)S_0 - (1+r)k\Delta_0 S_0, \Delta_0\}\,. \tag{30}$$

Clearly, the initial investment $V_0$ determining a fair option price can be found from

$$\frac{\partial R_0(S_0, V_0)}{\partial V_0} = 0\,. \tag{31}$$

## 5. Stochastic Volatility with Uncertainty: Adaptive Control

In this section we illustrate the usefulness of the multistage character of a stochastic optimization approach by considering the case in which the volatility is a stochastic variable and there is some uncertainty about it. We assume that the standard deviation (volatility)

$$\sigma_n = \int (\xi - \langle\xi\rangle)^2 \rho_n(\xi) d\xi \tag{32}$$

is a random sequence ($\langle\xi\rangle$ is a mean value).

For simplicity let us consider the case when the random variables $\sigma_n$ are independent from state to state and all have the same statistical properties, namely, $\sigma$ may have only two values $\sigma_1$ and $\sigma_2$ such that

$$\sigma = \begin{cases} \sigma_1 & \text{with the probability } p\,, \\ \sigma_2 & \text{with the proabability } 1-p\,, \end{cases} \qquad \sigma_1 > \sigma_2 \tag{33}$$

where the probability $p$ is not known in advance. The idea is to use an adaptive decision process [26] by which the uncertainty regarding $p$ can be reduced by the information gathered from the observation/calculation of current volatility. The basic idea is that the more decisions an investor makes, the more knowledge he gains about the stochastic dynamics of share prices, namely, volatility, and the better his subsequent decisions regarding trading strategy can become.

With enough information from the past history, the investor is supposed to have *a priori* probability density function $q = q(p)$ for $p$. We assume that from observation of the current state of volatility the investor can revise his subjective probability density function (pdf) $q(p)$. This is a basic idea of an adaptive or learning process. *A posteriori* probability density function $Q(p)$ can be determined as follows

$$Q(p) = \begin{cases} Q_1[q(p)] & \text{if } \sigma_1 \text{ occurs}\,, \\ Q_2[q(p)] & \text{if } \sigma_2 \text{ occurs}\,. \end{cases} \tag{34}$$

In particular, one can adopt the following rule [26]

$$Q_1 = \frac{pq(p)}{\langle p\rangle}\,, \qquad Q_2 = \frac{(1-p)q(p)}{1-\langle p\rangle}\,, \qquad \langle p\rangle = \int_0^1 pq(p)dp\,. \tag{35}$$

Let us introduce the minimal risk $R_N$ that can be achieved by starting from the arbitrary initial state $S_0, V_0$ with *a priori* probability density function $q(p)$ (no transaction costs)

$$R_N(S_0, V_0; q(p)) = \min_{\Delta_0\cdots\Delta_{N-1}} E\{(P_X(S_N) - V_N)^2\}\,, \tag{36}$$

then for one-stage process ($N = 1$)

$$\begin{aligned} &R_1(S_0, V_0; q(p)) \\ &\quad = \langle p\rangle \min_{\Delta_0} \int \{P_X(S_0 + \xi S_0) - (1+r)V_0 - \Delta_0(\xi - r)\}^2 \rho(\xi, \sigma_1)d\xi \\ &\qquad + (1 - \langle p\rangle) \min_{\Delta_0} \left\{ \int P_X(S_0 + \xi S_0) - (1+r)V_0 + \Delta_0(\xi - r)S_0 \right\}^2 \rho(\xi, \sigma_2)d\xi\,, \end{aligned} \tag{37}$$

where $\langle p\rangle$ is determined in (35). The recurrence relation for the dynamic programming solution for $N \geq 2$ is

$$\begin{aligned} R_N(S_0, V_0; q(p)) = {} & \langle p\rangle \min_{\Delta_0} \int R_{N-1}(S_0 + \xi S_0, (1+r)V_0 - \Delta_0(\xi - r); Q_1(P))\rho(\xi, \sigma_1)d\xi \\ & + (1 - \langle p\rangle) \min_{\Delta_0} \int R_{N-1}(S_0 + \xi S_0, (1+r)V_0 \\ & + \Delta_0(\xi - r)S_0; Q_2(p))\rho(\xi, \sigma_2)d\xi\,. \end{aligned} \tag{38}$$

This is the functional equation for the solution of the random volatility adaptive problem. It can be simplified if we assume a particular adaptive procedure to revise *a priori* probability density function $q(p)$. In particular, for the transformation (35), after $m+n$ stages in which $m$ times $\sigma_1$ and $n$ times $\sigma_2$ are observed, *a priori* pdf $q(p)$ is transformed into

$$\frac{p^m(1-p)^n q(p)}{\int_0^1 p^m(1-p)^n q(p)dp}\,.$$

As a result of this transformation, instead of pdf $q(p)$ one can use the numbers $m$ and $n$. It should be noted that the resulting functional equations are of higher dimensionality that for an equivalent stochastic process without adaptation. We believe that the stochastic optimization approach involving adaptive processes considered here is a promising way to deal with situations when there is an uncertainty about future security price and its volatility [12–15].

## 6. Forecasting Process

In this section we consider the case in which the investor can make an accurate prediction at the beginning of the period [n-1,n) that the value of the random parameter $\xi_n$ (see (1)) will be distributed in accordance with a particular probability density function among the set of given functions [27]. For simplicity we assume that the forecast, can take only the values 1 and 2. If the forecast is 1, then the random parameter $\xi_n$ is distributed in accordance with $\rho_n^{(1)}(\xi)$, if the forecast is 2, then the pdf for $\xi_n$ is $\rho_n^{(2)}(\xi)$. Of course we need to specify *a priori* probability for the forecast itself. We assume that at the time $n$ the forecast 1 occurs with probability $p_n$, and the forecast 2 occurs with the probability the probability $1-p_n$. The forecasting process can be described by the simple recurrence relation

$$Z_{n+1} = Y_n\,, \tag{39}$$

where $Y_n$ can take only two values 1 and 2 with corresponding probabilities $p_{n+1}$ and $1-p_{n+1}$.

Let us denote by $R_n(S_n, V_n, Z_n)$ the minimal risk for an $(N-n)$ stage investor's problem starting with the portfolio $V_n$ the security price $S_n$ and the forecast $Z_n$ at time $n$ and ending at time $N$.

$$R_n(S_n, V_n, Z_n) = \min_{\Delta_n,\ldots,\Delta_{N-1}} E\{(P_X(S_N) - V_N)^2\}\,, \tag{40}$$

where E is the expectation operator over all $\xi_n$ and $Y_n$.

In this situation the stochastic optimization algorithm takes the following form

$$\begin{aligned} R_n(S_n, V_n, Z_n) = p_{n+1} \min_{\Delta_n} E_{\xi_n}\{R_{n+1}(S_n + \xi_n S_n, (1+r)V_n \\ + \Delta_n(\xi_n - r)S_n, 1)\} + (1-p_{n+1}) \min_{\Delta_n} E_{\xi_n}\{R_{n+1}(S_n \\ + \xi_n S_n, (1+r)V_n + \Delta_n(\xi_n - r)S_n, 2)\}\,, \end{aligned} \tag{41}$$

where the average procedure $E_{\xi_n}$ is taken in accordance with the probability density function $p^{(Z_n)}(\xi)$.

Clearly this dynamic programming functional equation is analytically insolvable, but it is well suited to numerical solution. In order to solve (41) it is necessary to specify $R_N$ which is

$$R_N(S_N, V_N, Z_N) = (P_X(S_N) - V_N)^2 \,. \tag{42}$$

It follows from (41) that the optimal trading strategy at time is a function of the current value of share price $S_n$, portfolio $V_n$, and the current forecast $Z_n$, that is, $\Delta_n^* = \Delta_n^*(S_n, V_n, Z_n)$.

## 7. Analytical — Numerical Simulation.

The stochastic optimization problems can be solved analytically only in the simplest cases, for example, an one step model or a linear-quadratic problem. To this end we have developed a software package for the option pricing in an incomplete market based on the stochastic optimization procedure describe above. It consists of two parts: analytical and numerical. The analytical part has been designed by means of the computer-algebra package Maple, the numerical part has been written in C. The main feature of this approach is a flexible combination of analytical and numerical procedures. Namely, the solution of Bellman Eq. (21) has been constructed analytically by Maple in the form of recurrence equations.

### 7.1. *Single step option pricing model*

As an introduction to multistep systems, let us first consider a simple one step model. The dynamic of the stock price is governed by $S_1 = (1+\xi)S$, where $\xi$ is a random variable. The variation in the value of portfolio for one step can be written as $V_1 = (r+1)V + \Delta\eta S$, $\eta = \xi - r$. The option payoff is $P = \max(S_1 - X, 0)$, where $X$ is the exercise price. The expectation value

$$E\{(P - (r+1)V - \Delta\eta S)^2\} \tag{43}$$

should be minimized as a measure of risk. In (43) the expectation is taken with respect to the distribution of the random variable $\xi$. After averaging, the following expression for the residual risk can be obtained

$$\begin{aligned} R = \min_{V,\Delta}\{&(r+1)^2V^2 + 2(r+1)E(\eta)SV\Delta + E(\eta^2)S^2\Delta^2 \\ &-2(r+1)E(P)V - 2E(P\eta)S\Delta + E(P^2)\}\,. \end{aligned} \tag{44}$$

After minimization one can get the option price

$$C = V = \frac{E(P)E(\eta^2) - E(\eta)E(P\eta)}{(r+1)Var(\eta)}\,, \tag{45}$$

the optimal trading strategy

$$\Delta = \frac{E(P\eta) - E(P)E(\eta)}{Var(\eta)S}\,, \tag{46}$$

and residual risk

$$R = \frac{E(P^2)Var(\eta) - E^2(P\eta) + 2E(P\eta)E(P)E(\eta) - E^2(P)E(\eta^2)}{Var(\eta)}. \tag{47}$$

These expressions give the general solution of the single step problem for an arbitrary $\xi$. As an example, let us consider the binominal tree model. The random variable $\xi$ has the following properties:

$$\xi = \begin{cases} u & \text{with probability } \; p\,, \\ d & \text{with probability } \; 1-p\,. \end{cases} \tag{48}$$

Expectation in the formulas (45)-(47) for the binomial model can be written as

$$\begin{gathered} E(\eta) = (d-r)(1-p) + (u-r)p\,, \quad E(P) = C_1(1-p) + C_2 p\,, \\ E(\eta^2) = (d-r)^2(1-p) + (u-r)^2 p\,, \quad E(P^2) = C_1^2(1-p) + C_2^2 p\,, \\ E(\eta P) = C_1(d-r)(1-p) + C_2(u-r)p\,. \end{gathered} \tag{49}$$

By substituting (49) in (45)–(47) we obtain the option price

$$C = \frac{(r-d)C_2 + (u-r)C_1}{(1+r)(u-d)} \tag{50}$$

and optimal trading strategy

$$\Delta = \frac{C_2 - C_1}{S(u-d)}. \tag{51}$$

The option price and trading strategy are independent of probability $p$, while the residual risk for the binomial tree model is equal to zero. It is clear that the expressions (50) and (51) are the same as the well-known Cox, Ross and Rubinstein formulae [2].

### 7.2. *Multistep option pricing model*

The multistep model can be explored using the analytical-numerical software package. The stochastic optimization problem may be solved exactly by using computer-algebra package Maple for the arbitrary random sequence $\xi_n$ (see (1)).

Consider the share price $S(t)$ dynamics in the form

$$dS/S = \mu(S,t)dt + \sigma(S,t)dW\,. \tag{52}$$

To simulate $S(t)$ for which the drift and volatility are the functions of $S(t)$ and time $t$ we use a trinomial tree (TT) pricing model. First, we need to establish the relationship between the random process governed by (52) and a discrete-time version of the price dynamic (1). We denote by $N$, the number of trading dates, and $T$, the time to maturity. Let us suppose that trading occurs only at equidistant moments of time $\{0, t_1, \ldots, t_{N-1}, T\}$ with time step $\Delta t = t_{n+1} - t_n = T/N$. The dynamics of the stock price is then governed by $S_{n+1} = (1 + \xi_n)S_n$, where $\xi_n$ has the following properties:

$$\xi_n = \begin{cases} u & \text{with probability } \; p_1 \\ m & \text{with probability } \; p_2 \\ d & \text{with probability } \; 1 - p_1 - p_2 \end{cases} \tag{53}$$

such that $-1 < d < m < u$.

The discrete-time version of (52) can be written in the form

$$(S_{n+1} - S_n)/S_n = \mu(S_n, t_n)\Delta t + \sigma(S_n, t_n)\varepsilon\sqrt{\Delta t}\,, \tag{54}$$

where $\varepsilon$ is the random variable with standardized normal distribution.

It is natural to assume that the relative expected return and its variance are the same for both processes (53), and (54), that is

$$\mu\Delta t = E\{\xi_n\}\,, \tag{55}$$

$$\sigma^2\Delta t = E\{\xi_n^2\} - (E\{\xi_n\})^2\,. \tag{56}$$

The Eqs. (55) and (56) allow us to find the expressions for $\mu$ and $\sigma$ in terms of $u, m, d, p_1, p_2$ and $\Delta t$:

$$\mu\Delta t = p_1 u + p_2 m + (1 - p_1 - p_2)d\,, \tag{57}$$

$$\sigma^2\Delta t = p_1 u^2 + p_2 d^2 + (1 - p_1 - p_2)d^2 - (\mu\Delta t)^2\,. \tag{58}$$

Here we consider the so called recombinant trinomial tree [36]. This tree has $2n+1$ nodes at time $t_n$. The necessary and sufficient condition for the tree to recombine is

$$(1 + u)(1 + d) = (1 + m)^2\,. \tag{59}$$

Summarising, we have two probabilistic Eqs. (57) and (58) and one geometric Eq. (59) for the six parameters $u, m, d, p_1, p_2$ and $\Delta t$. It is traditional to allow the time step to remain free and we use two residual degrees of freedom to model the random process (52) for which the local relative expected return and variance are dependent on the asset price and time.

First of all we solve the systems of Eqs. (57) and (58) with respect to $p_1$ and $p_2$. The solutions are

$$p_1 = (md + \sigma^2\Delta t + (\mu\Delta t)^2 - (m + d)\mu\Delta t)/(u - m)/(u - d)\,, \tag{60}$$

$$p_2 = (ud + \sigma^2\Delta t + (\mu\Delta t)^2 - (d + u)\mu\Delta t)/(u - d)/(u - m)\,. \tag{61}$$

We define geometrical tree parameters $u, m, d$ from the following conditions:

$$u = \tilde{\sigma}\sqrt{\Delta t} + \tilde{\mu}\Delta t,\,, \tag{62}$$

$$d = -\tilde{\sigma}\sqrt{\Delta t} + \tilde{\mu}\Delta t\,, \tag{63}$$

$$m = \sqrt{(1 + u)(1 + d)} - 1\,, \tag{64}$$

where $\tilde{\sigma}$ and $\tilde{\mu}$ are new parameters. These parameters can be defined from the stability analysis so that the probabilities $p_1$ and $p_2$ are non-negative and less then one. By choosing the parameters $\tilde{\sigma}$ and $\tilde{\mu}$, the time step $\Delta t$, extreme values of drift parameter $\mu_{\max}$, $\mu_{\min}$ and volatility $\sigma_{\max}$, $\sigma_{\min}$ we determine the geometrical tree parameters $u, m$, and $d$ from (62)–(64). After that we can calculate the probabilities $p_1$ and $p_2$ in each node of the tree according to (60) and (61). It should be noted that $p_1$ and $p_2$ for (54) are the functions of the stock price and time

$$p_1^{i,n} = p_1(S_n^i, t_n)\,, \quad p_2^{i,n} = p_2(S_n^i, t_n)\,, \tag{65}$$

where $n$ is the time variable and $i$ is the height of the tree. The details of this procedure can be found in [36].

### 7.3. *Results of numerical simulations*

We assume here that the opening price of the underlying asset is \$100, the strike is \$90, the maturity of the option is 90 days, the interest rate is 0.05, the volatility for the discrete-time Black–Scholes (BS) model is 0.3, the tree drift parameter $\tilde{\mu}$ is 0.05 and the tree volatility parameter $\tilde{\sigma}$ is 0.55. In the discretetime world the BS model corresponds to the binomial tree. We can get a binomial tree in the framework of trinomial tree model by assuming $p_2 = 0$. It is well-known that for the BS model market is complete. This means that there exists a dynamic trading strategy in the underlying asset that exactly replicates the derivative contract payoff. The BS option price does not depend upon the expected return $\mu$ and probability $p_1$, while the residual risk is equal to zero. Here our purpose is to compare option prices calculated according to discrete-time BS pricing model and the trinomial tree (TT) pricing model based on the stochastic optimization techniques. We assume that the volatility is a function of the share price $\sigma = \sigma(S)$. The volatility profile implemented is shown in Fig. 1.

This volatility profile reflects the "rational behavior" of market-makers. If the market goes up then the volatility decreases, if the stock declines then the volatility grows. In each variant we fix all the basic parameters of the model except one, and represent the difference between the options prices for BS and TT models as the

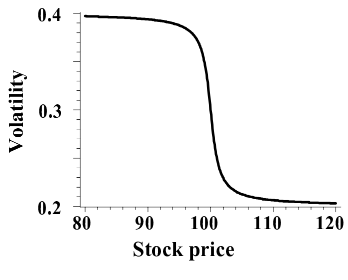


Fig. 1. Volatility as a function of stock price.

function of this (not fixed) variable. For the TT model we also evaluate the risk, $\sqrt{R}$, as a function of the same variable.

Figure 2 shows the typical option price and the residual risk convergence pattern for the TT pricing model as the number of tree nodes increases. The horizontal line corresponds to the numerical solution to the TT model with 2000 steps.

In Figs. 3 and 4 we show the difference between TT and discrete-time BS option price and residual risk as functions of the opening stock price $S_0$ and the strike price $X$ respectively.

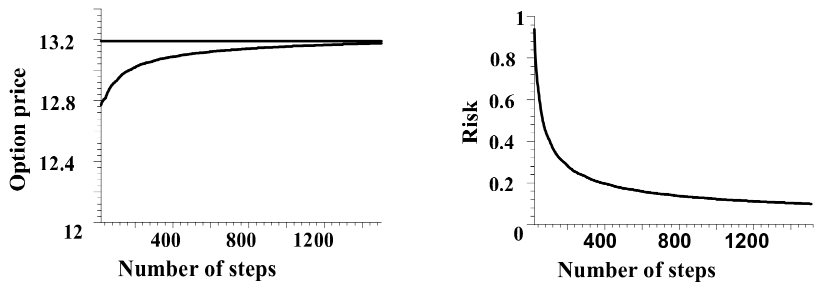


Fig. 2. Typical convergence pattern for TT pricing model.

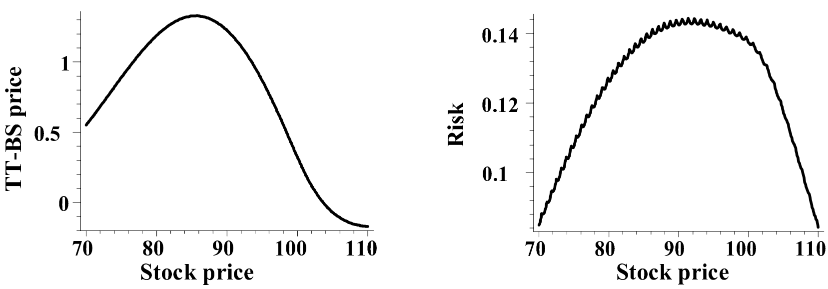


Fig. 3. The difference between TT and discrete-time BS option prices and residual risk as functions of the opening stock price.

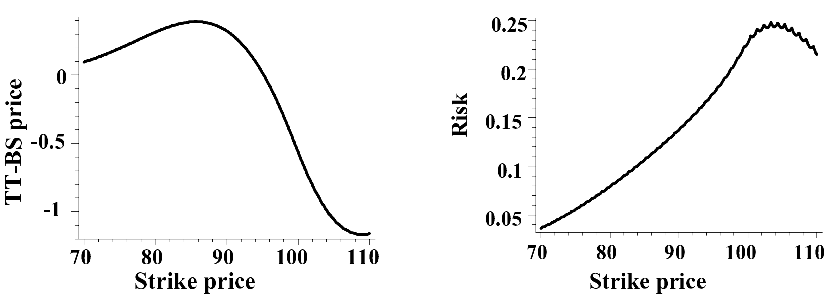


Fig. 4. The difference between TT and discrete-time BS option prices and residual risk as functions of the strike price.

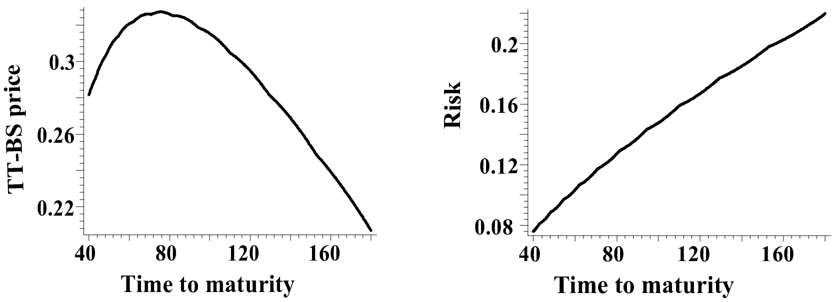


Fig. 5. TT-BS option price and risk as a function of time to expiry.

From Fig. 4 it can been seen that the TT pricing model with a volatility profile traced in Fig. 1 gives higher option prices with respect to BS prices in out-the-money and at-the-money zones. Within the in-the-money region the TT prices are less than BS prices. However, the relative difference is rather small in all regions. Note that the value of residual risk becomes very small for the far out-of-the-money region. The same pattern of the residual risk was observed in [23].

In Fig. 5 we show the TT-BS price difference and residual risk for different time intervals to expiration.

The basic advantage of using the trinomial model over the classical binomial model is that it allows us to simulate the stock price dynamics using the drifts and volatilities that depend on the value of the underlying security. But TT models are much more difficult to work with. In the case of the trinomial tree there is no dynamic trading strategy in the underlying security that exactly replicates the derivative contract payoff and as a result there exists a risk that should be minimized by optimal hedging. However, the numerical simulations show that the trinomial model gives an option price that is only slightly different from that of the binomial model. This may explain why the binomial models (which are simple and robust) are so popular among practitioners.

## 8. Summary

To conclude, an effective algorithm based on the discrete stochastic optimization approach has been presented that gives the option price and optimal trading strategy for the incomplete market in which the risk incurred by selling an option cannot be completely hedged by dynamic trading. We illustrate the usefulness of the multistage optimization procedure by considering the effects of transaction costs, stochastic volatility with uncertainty, adaptive and forecasting processes. It should be noted that the usual way to cope with our ignorance of future security prices is to introduce the random processes with *known* statistical characteristics. However it is very easy to imagine a situation when so little is known about future prices in advance that it is simply impossible to suggest the exact statistics. We believe that the stochastic optimization approach involving adaptive and forecasting processes considered here is a promising way to deal with such situations.

A program package for the option pricing in an incomplete market has been developed. The package consists of two parts: analytical (Maple) and numerical (C). The stochastic optimization problem has been solved analytically by using Maple in the form of recurrent equations. The numerical part consists of rather simple calculations of recurrent equations.

There are several future directions to explore by using the method presented here. First, one may study the case with imperfect state information regarding the asset prices by using a stochastic game approach. Also, one can study various adaptive and forecast control problems. It should be noted that the preliminary work done here might be of big practical importance and it therefore merits further investigation including the computational aspect of our formalism.

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