Non-Markovian random processes and traveling fronts in a reaction-transport system with memory and long-range interactions

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The problem of finding the propagation rate for traveling waves in reaction-transport systems with memory and long-range interactions has been considered. Our approach makes use of the generalized master equation with logistic growth, hyperbolic scaling, and Hamilton-Jacobi theory. We consider the case when the waiting-time distribution for the underlying microscopic random walk is modeled by the family of gamma distributions, which in turn leads to non-Markovian random processes and corresponding memory effects on mesoscopic scales. We derive formulas that enable us to determine the front propagation rate and understand how the memory and long-range interactions influence the propagation rate for traveling fronts. Several examples involving the Gaussian and discrete distributions for jump densities are presented.

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I. INTRODUCTION

The problem of calculating the propagation rate for traveling waves in reaction-transport systems with an unstable state has attracted widespread interest from the scientific community in recent years. This interest is due to the wide variety of problems, examples include population growth and dispersion, the spread of epidemics, combustion waves, and magnetic front propagation, etc. [1–8]. The simplest model used is the Fisher-Kolmogorov-Petrovskii-Piskunov (FKPP) equation $\partial n/\partial t = D\partial^2 n/\partial x^2 + Un(1-n)$, where D is the diffusion coefficient and U is the reaction rate parameter [1–3]. However, there exist certain deficiencies in the above model, in particular as $U\to\infty$ the propagation rate for traveling waves $u=2(UD)^{1/2}$ becomes infinite, and this clearly contradicts the physical fact that the rate u should be finite. This shortcoming of the FKPP equation is due to the diffusion approximation for the transport process. Several studies have focused on modifications to the classical diffusion model that include more information about the particles dynamics on a microscopic level [4–7].

Very recently there have been some important developments in the theory of wave propagation for the integro-differential equations and integro-difference equations involving a space integral [4,6,7,9]. These equations provide more realistic models for various wave phenomena in physics, chemistry, biology, etc. Various examples of the space integral terms describing long-range interactions and their physical and biological meanings can be found in an excellent book (p. 481 of Ref. [1]). The macroscopic transport process comes from the overall effect of many particles (turbulent eddies, bacteria, animals, etc.) performing very complex random movements on a microscopic level. The classical diffusion term in the FKPP equation is just an approximation for this transport in the long-time large-distance parabolic scaling limit. In other words, microscopic random walks are modeled by Brownian motion that has no jumps and inertia or characteristic relaxation time. It has been shown [4–9] that in general the diffusion approximation for a transport process is not correct for problems involving traveling waves for which the appropriate scaling must be hyperbolic. The basic idea is that production term described by the logistic growth $Un(1-n)$ is very sensitive to the tails of a concentration profile, while these tails are typically "nonuniversal," "nondiffusional" and dependent on the microscopic details of the transport processes. While, on average, transport processes may behave diffusively, unstable media are more affected by the weak tails of the transport process. As a result, the macroscopic dynamics of fronts propagating into an unstable state of the reaction transport system ought to be dependant upon the particular random walk model underlying the transport process. The detailed discussion of this idea can be found in Refs. [6,7]. From a practical point of view, this is a very important conclusion showing simple models based on reaction-diffusion ideas used in physics, mathematical biology, etc., do not work properly in general. Hence it is desirable to extend these results by considering more realistic models for the transport processes based on integro-differential and integro- difference equations.

It would be interesting now to consider the integrodifferential equations with the time integral as well. An advantage being that this will allow us to take into account the memory effects associated with non-Markovian random processes. It is clear that due to non-Markovian character of the microscopic movements of animals, bacteria, etc., their random walks cannot be approximated by Brownian motion in general. Memory effect is a significant feature in many areas of physics, chemistry, and biology, but may often be ignored through the difficulties of how to deal with it [1,8,10]. In fact, only Markov random processes have been considered in Refs. [6,7]. The main purpose of this paper is to find out how non-Markovian random processes with long-range interactions and associated memory effects influence the propagation of traveling waves into an unstable state of a reaction-transport system.

II. MESOSCOPIC DESCRIPTION: GENERALIZED MASTER EQUATION WITH LOGISTIC GROWTH

To take into account memory effects and long-range interactions, we consider the following generalized master equation with logistic growth:

$$\frac{\partial n}{\partial t}=\int_0^t\int_{-\infty}^{\infty}[K(x,z,t-s)n(s,z)-K(z,x,t-s)n(s,x)]dzds+Un(1-n) \tag{1}$$

with the frontlike initial condition

$$n(0,x)=\theta(x), \tag{2}$$

where $\theta(x)$ is the Heaviside function: $\theta(x)=1$ for $x\leqslant 0$ and $\theta(x)=0$ for $x>0$.

Equation (1) can be considered as a natural generalization of the FKPP equation in the case when the memory and long-range interactions are taken into account. In what follows we assume that $n(t,x)$ is the mesoscopic concentration of particles at position x at time t. The problem here is to give an explicit expression for the kernel $K(x,z,t-s)$. In this paper we restrict ourselves to the case of the factorized kernel only,

$$K(x,z,t-s)=\alpha(t-s)\varphi(x-z). \tag{3}$$

In particular, when $K(x,z,t-s)=2D\delta(t-s)[1/2\delta(z-1)+1/2\delta(z+1)]$ one can get a discrete version of the FKPP equation, namely,

$$\frac{\partial n}{\partial t}=D[n(t,x+1)-2n(t,x)+n(t,x-1)]+Un(1-n). \tag{4}$$

The aims of this paper are (i) to find the propagation rate for the traveling waves described by the integro-differential equation (1) with initial condition (2) and (ii) to find the connection between the *mesoscopic* description of the particles concentration in terms of the integral operator in Eq. (1) and the *microscopic* random walk of one particle. In particular, we are going to consider the case when the waiting-time distribution for the underlying random walk is modeled by the family of gamma distributions, which in turn leads to non-Markovian random processes and corresponding memory effects on a mesoscopic scale. To the authors knowledge, this paper is the first attempt to take into account both memory effects and long-range interactions in the problem of wave propagation into an unstable state of a reaction-transport system.

III. MICROSCOPIC DESCRIPTION

A. Underlying microscopic random walk model

In this section we discuss the underlying *microscopic* random walk models corresponding to the transport operator in Eq. (1). The key question when using the integro-differential equation (1) with the kernel (3) to model *mesoscopic* dynamics of particles concentration is to determine the functions $\alpha(t-s)$ and $\varphi(z)$ in terms of the statistical characteristics of the underlying random walk. For this reason let us consider the following one dimensional random process. Suppose a particle starts from some initial position, where it remains for some random time before performing a jump of random length, it remains here for some random time before performing another jump of random length and so on. Let $\psi(t)$ be the probability density function (PDF) of the waiting time between successive jumps. Let $\varphi(z)$ be the probability density function of jump size, which we assume to be independent of time. The problem of this kind, when the growth is absent $(U=0)$, was dealt with extensively in literature (see, for example, Ref. [13]), and is often termed the continuous-time random walk (CTRW). If we apply the Fourier transform in space and Laplace transform in time for the corresponding probability density function $n(t,x)$,

$$\tilde{n}(E,k)=\int_{-\infty}^{\infty}\int_0^{\infty}e^{-Et+ikx}n(t,x)dtdx, \tag{5}$$

then [12,13]

$$\tilde{n}(E,k)=\frac{[1-\hat{\psi}(E)]\tilde{n}(0,k)}{E[1-\hat{\psi}(E)\tilde{\varphi}(k)]}, \tag{6}$$

where $\hat{\psi}(E)$ is the Laplace transform of $\psi(t)$, and $\tilde{\varphi}(k)$ is the Fourier transform of $\varphi(z)$. For the CTRW there is an associated integro-differential equation of the form [12,13]

$$\frac{\partial n(t,x)}{\partial t}=\int_0^t\alpha(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds, \tag{7}$$

where $n(t,x)$ is defined to be the probability of a particle being at a site x at time t. Recall the well known Kenkre, Montroll, and Shlesinger result, which states that there exists a relationship between the memory kernel $\alpha(t)$ in Eq. (7) and the waiting time PDF $\psi(t)$, namely [12],

$$\hat{\psi}(E)=\frac{\hat{\alpha}(E)}{E+\hat{\alpha}(E)} \quad \text{or} \quad \hat{\alpha}(E)=\frac{E\hat{\psi}(E)}{1-\hat{\psi}(E)}, \tag{8}$$

where $\hat{\alpha}(E)$ is the Laplace transform of $\alpha(t)$. It is easy to see that in the case $U=0$ Eq. (7) is equivalent to Eq. (1) with the kernel (3).

Following Ref. [14] one can derive the following equation for the probability density $n(t,x)$ with the linear logistic growth $(U\neq 0)$:

$$n(t,x)=\Psi(t)n(0,x)+\int_0^t\int_{-\infty}^{\infty}\psi(t-s)n(s,x+z)\varphi(z)dzds+U\int_0^t\Psi(t-s)n(s,x)ds, \tag{9}$$

where

$$\Psi(t)=1-\int_0^t \psi(s)ds \tag{10}$$

is the probability that the particle does not move up to time t [14]. It should be noted that Eq. (8) still holds, and Eq. (9) is equivalent to

$$\frac{\partial n(t,x)}{\partial t}=\int_0^t \alpha(t-s)\left[\int_{-\infty}^{\infty} n(s,x+z)\varphi(z)dz-n(s,x)\right]ds + Un(t,x). \tag{11}$$

B. The family of gamma distributions

To model the waiting time density $\psi(t)$ we use here the family of gamma distributions with parameters m and λ,

$$\Gamma_{m,\lambda}(t)=\frac{\lambda^m t^{m-1}e^{-\lambda t}}{\Gamma(m)}, \quad m\in\mathbb{N}, \tag{12}$$

where $\Gamma(m)$ is the gamma function. Our motivation for such a choice of the waiting time density $\psi(t)$ is that when $m=1$ we have the exponential pdf $[\psi(t)=\lambda e^{-\lambda t}]$, the only "memoryless" type PDF, whereas for all other $m\in\mathbb{N}$ we have non-Markovian dynamics for the underlying random walk. Our aim is to investigate how the front propagation rate changes through the introduction of non-Markovian effects.

When $U=0$, one can derive the following partial differential equation involving time derivatives up to order m (see Appendix A):

$$(D_t+\lambda)^m n(t,x)=\lambda^m\int_{-\infty}^{\infty} n(t,x+z)\varphi(z)dz, \tag{13}$$

where D_t is the partial derivative operator with respect to t, i.e., $D_t n(t,x)=\partial n/\partial t$, $D_t^2 n(t,x)=\partial^2 n/\partial t^2$, etc.

When $U\neq 0$ and $m=1$, one can get from Eqs. (8) and (12) that $\hat{\psi}(E)=\lambda/(\lambda+E)$, hence, $\hat{\alpha}(E)=\lambda$ that is $\alpha(t-s)=\lambda\,\delta(t-s)$. Substitution of this expression into Eq. (11) gives a classical Feller-Kolmogorov equation with a linear growth [14],

$$\frac{\partial n(t,x)}{\partial t}=\lambda\left[\int_{-\infty}^{\infty} n(t,x+z)\varphi(z)dz-n(t,x)\right]+Un(t,x). \tag{14}$$

The case $m=2$ corresponds to a non-Markovian random walk. One can find that the Laplace transform $\hat{\psi}(E)=\lambda^2/(\lambda+E)^2$, hence $\hat{\alpha}(E)=\lambda^2/(E+2\lambda)$, that is,

$$\alpha(t-s)=\lambda^2\exp[-2\lambda(t-s)]. \tag{15}$$

Equation (11) takes the form

$$\frac{\partial n(t,x)}{\partial t}=\lambda^2\int_0^t e^{-2\lambda(t-s)}\left[\int_{-\infty}^{\infty} n(s,x+z)\varphi(z)dz - n(s,x)\right]ds+Un(t,x). \tag{16}$$

IV. PROPAGATION RATE FOR TRAVELING WAVES

In this section, we consider the problem of calculating the propagation rate of traveling waves for the integro-differential equation (1) when the waiting time PDF is given by a member of the family of gamma distributions (12). The frontlike initial condition (2) ensures the minimal speed of propagation [1]. The standard way to deal with the above problem is to find a traveling wave solution in the form $n(t,x)=f(x-ut)$. Here we consider the problem in the hydrodynamic limit where the problem of wave propagation into an unstable state can be reduced to a problem of the dynamics of the reaction front [2,6,7,11]. If we make a hyperbolic scaling $t\to t/\varepsilon$, $x\to x/\varepsilon$, where ε is a small parameter we encounter the Cauchy problem for a rescaled particles concentration $n^\varepsilon(t,x)=n(t/\varepsilon,x/\varepsilon)$. We expect that $n^\varepsilon(t,x)$ tends to a step function as $\varepsilon\to 0$ [2,11]. The idea now is to derive the eikonal equation governing the position of front. If we first replace $n^\varepsilon(t,x)$ by an auxiliary field $G(t,x)\geqslant 0$ by using the exponential transformation $n^\varepsilon(t,x)=\exp[-G(t,x)/\varepsilon]$, then expanding to leading order we can deduce the Hamilton-Jacobi equation $\partial G/\partial t+H(\partial G/\partial x)=0$. When the reaction rate parameter U is independent of space, one can find that $G(t,x)=px-H(p)t$, which we recognize as the action functional of a free particle. Taking into account $x=ut$ we find from $G(t,x(t))=0$ the propagation rate u [6,7],

$$u=\frac{\partial H}{\partial p}, \quad pu=H(p). \tag{17}$$

These two equations together with the appropriate Hamiltonian $H(p)$ give the complete solution to the problem of front propagation [6,7]. Recall that the Hamiltonian for the FKPP equation is given by $H(p)=Dp^2+U$ [2,11] thus from Eq. (17) we can obtain the classical propagation rate $u=2\sqrt{DU}$. Let us note that the same results can be derived by using marginal stability analysis [3].

Now we are in a position to find the Hamiltonian function $H(p)$ corresponding to the integro-differential equation (1) with the kernel (3). It is well known that the rate u at which the traveling wave propagates into an unstable state depends on the leading edge of the scalar field profile where $n(t,x)\to 0$ [1–3]. Therefore to find u it is sufficient to consider the linearized version of Eq. (1). By similar methodology used to obtain Eq. (13), we can derive the equation for the particles concentration $n(t,x)$ with growth $(U\neq 0)$ (see Appendix B),

$$(D_t+\lambda)^m n(t,x)=\lambda^m\int_{-\infty}^{\infty} n(t,x+z)\varphi(z)dz + U\sum_{r=1}^{m}\binom{m}{r}\lambda^{m-r}(D_t)^{r-1}n(t,x). \tag{18}$$

After making a hyperbolic scaling $t\to t/\varepsilon$, $x\to x/\varepsilon$ we obtain for $n^\varepsilon(t,x)=n(t/\varepsilon,x/\varepsilon)$,

$$(\varepsilon D_t+\lambda)^m n^\varepsilon(t,x)=\lambda^m\int_{-\infty}^{\infty}n^\varepsilon(t,x+\varepsilon z)\varphi(z)dz + U\sum_{r=1}^{m}\binom{m}{r}\lambda^{m-r}(\varepsilon D_t)^{r-1}n^\varepsilon(t,x). \tag{19}$$

We seek a solution of Eq. (19) as $\varepsilon\to 0$ of the exponential form $n^\varepsilon(t,x)=\exp[-G(t,x)/\varepsilon]$. Substituting this expression into Eq. (19) we find that the leading order behavior as $\varepsilon\to 0$ is given by the Hamilton-Jacobi equation for $G(t,x)$,

$$\frac{\partial G}{\partial t}+H\left(\frac{\partial G}{\partial x}\right)=0, \tag{20}$$

where $H(p)$ represents the Hamiltonian associated with Eq. (19) and can be found from

$$[H(p)+\lambda]^m=\lambda^m f(p)+U\sum_{r=1}^{m}\binom{m}{r}\lambda^{m-r}[H(p)]^{r-1}, \tag{21}$$

where $f(p)=\int_{-\infty}^{\infty}e^{-zp}\varphi(z)dz$ is the bilateral Laplace transform of the PDF $\varphi(z)$, also known as the moment generating function. By using Eqs. (17) and (21), we can now find the propagation rate u when the waiting density $\psi(t)$ is given by a member of the family of gamma distributions $\Gamma_{m,\lambda}(t)$, $m\in\mathbb{N}$.

Remark. It is interesting to note that when the growth is absent, i.e., $U=0$, the Hamiltonian $H(p)$ can be shown to be

$$H(p)=\lambda[f(p)^{1/m}-1], \tag{22}$$

in particular, if we suppose that the jump density $\varphi(z)$ is given by the normal distribution $N(0,\sigma^2)$, then $f(p)=\exp(p^2\sigma^2/2)$ and so the Hamiltonian is now given by

$$H(p)=\lambda\left[\exp\left(\frac{p^2\sigma^2}{2m}\right)-1\right]. \tag{23}$$

Thus, the effect of changing the waiting density from $\Gamma_{1,\lambda}(t)$ to $\Gamma_{r,\lambda}(t)$ say, can be replicated by changing the jump density from $N(0,\sigma^2)$ to $N(0,\sigma^2/r)$.

A. Example 1 (no memory)

As our first example let us consider the case when the waiting density $\psi(t)$ is of the exponential form, $\Gamma_{1,\lambda}(t)=\lambda e^{-\lambda t}$. Let us also suppose that the jump density $\varphi(z)$ is given by the normal distribution $N(0,\sigma^2)$. From Eq. (21) one can find the Hamiltonian

$$H(p)=\lambda\left[\exp\left(\frac{p^2\sigma^2}{2}\right)-1\right]+U. \tag{24}$$

If we denote the propagation rate by u_1 then

$$u_1=\frac{\partial H(p)}{\partial p}=\lambda\sigma^2 p\exp\left(\frac{p^2\sigma^2}{2}\right), \tag{25}$$

where the momentum p can be found from (17)

$$(1-p^2\sigma^2)\exp\left(\frac{p^2\sigma^2}{2}\right)=1-\frac{U}{\lambda}. \tag{26}$$

It follows from Eqs. (25) and (26) that the introduction of the long-range interactions without memory leads to the increase of the speed of traveling fronts (see Sec. IV C).

B. Example 2 (with memory effects)

Now consider the case when the waiting density $\psi(t)$ is given by $\Gamma_{2,\lambda}(t)=\lambda^2 te^{-\lambda t}$, and the jump density $\varphi(z)$ is given by the normal distribution $N(0,\sigma^2)$. From Eq. (21) we have a quadratic equation for $H(p)$,

$$[H(p)+\lambda]^2=\lambda^2 e^{p^2\sigma^2/2}+U(H(p)+2\lambda).$$

Solving this equation,

$$H(p)=\frac{U}{2}-\lambda+\sqrt{\frac{U^2}{4}+U\lambda+\lambda^2,e^{p^2\sigma^2/2}}$$

we can find the propagation rate u_2,

$$u_2=\frac{\partial H(p)}{\partial p}=\frac{\lambda^2\sigma^2 pe^{p^2\sigma^2/2}}{2\sqrt{\frac{U^2}{4}+U\lambda+\lambda^2 e^{p^2\sigma^2/2}}}, \tag{27}$$

where p has to be found from Eq. (17),

$$\frac{\sigma^2p^2e^{p^2\sigma^2/2}}{2\sqrt{\frac{U^2}{4\lambda^2}+\frac{U}{\lambda}+e^{p^2\sigma^2/2}}}=\frac{U}{2\lambda}-1+\sqrt{\frac{U^2}{4\lambda^2}+\frac{U}{\lambda}+e^{p^2\sigma^2/2}}. \tag{28}$$

One can show that the introduction of memory effects leads to a decrease of the propagation rate (see the following section).

C. Comparison with the FKPP equation

Here we compare the propagation speeds generated by the waiting densities: $\Gamma_{1,\lambda}(t)$, $\Gamma_{2,\lambda}(t)$, and $\Gamma_{3,\lambda}(t)$, denoted by u_1, u_2, and u_3, respectively, and the FKPP propagation rate u,

$$u=2\sqrt{UD}, \tag{29}$$

where

$$D=\frac{\langle z^2\rangle}{2\tau},\quad \langle z^2\rangle=\int_{-\infty}^{\infty}z^2\varphi(z)dz \tag{30}$$

and τ is defined to be the mean time between successive jumps. For the waiting density $\Gamma_{m,\lambda}(t)$, the mean time τ is m/λ. It is convenient to write the FKPP propagation rate in the form

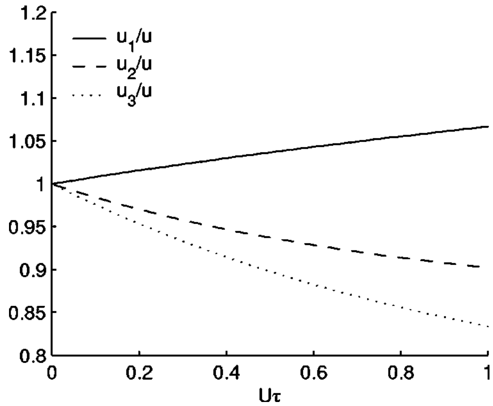

FIG. 1. Dependencies of the ratio u_m/u on $U\tau$ for various values of m when the jump density $\varphi(z)$ is $\frac{1}{2}\delta(t+a)+\frac{1}{2}\delta(t-a)$.

$$u=u_0\sqrt{2U\tau}, \quad u_0=\frac{\langle z^2\rangle^{1/2}}{\tau}, \tag{31}$$

where u_0 can be regarded as the characteristic speed of the random walk. It turns out that u_m/u depends only upon the parameter $U\tau$.

In Figs. 1 and 2, we show u_m/u plotted against $U\tau$ for various values of m when the jump density $\varphi(z)$ is given by (1) $\frac{1}{2}\delta(t+a)+\frac{1}{2}\delta(t-a)$ and (2) $N(0,\sigma^2)$, respectively. In both figures we see that for the case when $\psi(t)=\Gamma_{1,\lambda}(t)=\lambda\exp(-\lambda t)$ the propagation rate u_1 is greater than u, and the ratio u_1/u increases as $U\tau$ increases. This should be expected, since the FKPP equation implicitly involves a second order approximation for the jump density $\varphi(z)$, i.e., up to $\langle z^2\rangle/2$. In fact, if we suppose that $U\tau\ll 1$, then by writing (1) $f(p)=\cosh(ap)\approx 1+a^2p^2/2$ and (2) $f(p)=\exp(p^2\sigma^2/2)\approx 1+p^2\sigma^2/2$, one can show that the propagation rate u_1 is equal to the FKPP propagation rate u. However, for $U\tau\lesssim 1$ this approximation is not appropriate, and so we have to take into account the tails of $\varphi(z)$ which contribute to the increase in the propagation rate. From Fig. 1 we can conclude that, when the waiting density $\psi(t)$ is given by $\Gamma_{2,\lambda}(t)$ and $\Gamma_{3,\lambda}(t)$, with propagation rate u_2 and u_3 respectively, both u_2 and u_3 become slower than u as $U\tau$ increases. This difference explicitly demonstrates that the introduction of ''memory effects'' into our random walk model decreases the propagation rate of the traveling wave. In Fig. 2, we observe a similar phenomenon, with the exception that for $\psi(t)=\Gamma_{2,\lambda}(t)$ the propagation rate u_2 is still greater than u, whereas for $m>2$, the propagation rate decreases. From these two simple examples, we can conclude that although the introduction of memory effects and long-range interactions certainly affect the propagation rate of the traveling wave, it is not immediately clear as to whether this rate will be more or less than the propagation rate calculated by the FKPP equation.

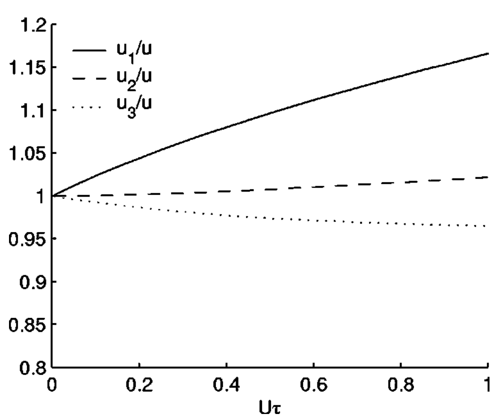

FIG. 2. Dependencies of the u_m/u on $U\tau$ for various values of m for the jump density $\varphi(z)$ given by the normal distribution $N(0,\sigma^2)$.

V. SUMMARY

In this paper we have investigated the problem of determining the propagation rate for traveling waves in a reaction-transport system with memory and long-range interactions. In particular, we have used the family of gamma distributions to model the waiting-time density in order to gain an understanding of how non-Markovian dynamics affects the behavior of the traveling fronts. Using a generalized master equation with logistic growth, hyperbolic scaling, and Hamilton-Jacobi theory we have derived formulas which enable us to determine the front propagation rate. By using two simple examples we have shown that for the case when the characteristic growth time U^{-1} is much larger than the mean time between successive jumps τ ($U\tau\ll 1$), the FKPP equation is an appropriate model in determining the propagation rate. However, when $U\tau\lessdot 1$ the inability of the FKPP model to take into account the tails of the jump density and memory effects induced by non-Markovian densities, leads to an overestimation/underestimation of the propagation rate. We have shown that it is not immediately clear as to whether the propagation rate will be greater or less than that found via the FKPP equation, and detailed study of both the jump density and waiting-time density is required before this conclusion can be made.

APPENDIX A: DERIVATION OF EQ. (13)

We write the generalized master equation in the form [13]

$$\frac{\partial n(t,x)}{\partial t}=\int_0^t \alpha(t-s)\left[\int_{-\infty}^{\infty} n(s,x+z)\varphi(z)dz-n(s,x)\right]ds. \tag{A1}$$

We know from Eq. (8) that the Laplace transform of $\alpha(t)$ is

$$\hat{\alpha}(E)=\frac{E\hat{\psi}(E)}{1-\hat{\psi}(E)}.$$

If we introduce the auxiliary function

$$\hat{f}(E)=\frac{\hat{\psi}(E)}{1-\hat{\psi}(E)},$$

then $\alpha(t)$ can be written in terms of the inverse Laplace transform L^{-1},

$$\alpha(t)=L^{-1}[\hat{\alpha}(E)]=\frac{d}{dt}\{L^{-1}[\hat{f}(E)]\}+f(0)\delta(t). \tag{A2}$$

Since $|\hat{\psi}(E)|<1$, one can write

$$\hat{f}(E)=\frac{\hat{\psi}(E)}{1-\hat{\psi}(E)}=\sum_{j=1}^{\infty}[\hat{\psi}(E)]^j. \tag{A3}$$

For the waiting time PDF $\psi(t)$, we choose a member of the gamma family of densities (12). The Laplace transform of $\Gamma_{m,\lambda}(t)$ is given by

$$\hat{\Gamma}_{m,\lambda}(E)=\left(\frac{\lambda}{\lambda+E}\right)^m. \tag{A4}$$

It follows from Eqs. (A3) and (A4) that

$$\begin{aligned} f(t)&=L^{-1}\left[\sum_{j=1}^{\infty}[\hat{\psi}(E)]^j\right]\\ &=L^{-1}\left[\sum_{j=1}^{\infty}\left(\frac{\lambda}{\lambda+E}\right)^{jm}\right]\\ &=\sum_{j=1}^{\infty}\Gamma_{jm,\lambda}(t) \end{aligned} \tag{A5}$$

and $f(0)=\lambda$, for $m=1$, while $f(0)=0$, for $m>1$. Note that

$$\frac{d^r\Gamma_{m,\lambda}(t)}{dt^r}=\lambda^r\sum_{j=0}^{r}\binom{r}{j}\Gamma_{m-j,\lambda}(t)(-1)^{r-j}, \quad r\leqslant m, \tag{A6}$$

where $\Gamma_0(t)=0$. Let us fix m, and define a new function $A_l(t)$ as an infinite sum of certain gamma densities, whereby the first term in the sum is $\Gamma_{l,\lambda}(t)$, the second $\Gamma_{l+m,\lambda}(t)$, the third $\Gamma_{l+2m,\lambda}(t)$, etc. From Eqs. (A5) and (A6) we have

$$f^{(r)}(t)=\lambda^r\sum_{j=0}^{r}\binom{r}{j}A_{m-j}(t)(-1)^{r-j}, \quad r\leqslant m \tag{A7}$$

and $f^{(r)}(0)=\lambda^m$ for $r=m-1$, while $f^{(r)}(0)=0$, for $r<m-1$. One can show that

$$\sum_{r=0}^{m}\binom{m}{r}\lambda^{m-r}f^{(r)}(t)=0. \tag{A8}$$

The case when $m=1$ has already been considered in Sec. III B. Let us further suppose that $m>1$, thus $f(0)=0$. It follows from Eqs. (A1) and (A2) that

$$\frac{\partial n}{\partial t}=\int_0^t f^{(1)}(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds.$$

Further partial differentiation with respect to time gives

$$\frac{\partial^r n}{\partial t^r}=\int_0^t f^{(r)}(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds, \quad r\leqslant m-1, \tag{A9}$$

$$\frac{\partial^m n}{\partial t^m}=\int_0^t f^{(m)}(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds +\lambda^m\left[\int_{-\infty}^{\infty}n(t,x+z)\varphi(z)dz-n(t,x)\right]. \tag{A10}$$

Now we obtain from Eqs. (A9) and (A10)

$$\sum_{r=0}^{m}\lambda^{m-r}\binom{m}{r}\frac{\partial^r n}{\partial t^r}=\lambda^m\int_{-\infty}^{\infty}n(t,x+z)\varphi(z)dz.$$

This equation can be rewritten as

$$(D_t+\lambda)^m n(t,x)=\lambda^m\int_{-\infty}^{\infty}n(t,x+z)\varphi(z)dz.$$

APPENDIX B: DERIVATION OF EQ. (18)

Our procedure here is the same as in Appendix A, with the exception that Eq. (A1) is modified through the addition of the linear logistic growth term

$$\frac{\partial n(t,x)}{\partial t}=\int_0^t \alpha(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds +Un(t,x). \tag{B1}$$

Thus Eqs. (A9) and (A10) become

$$\frac{\partial^r n}{\partial t^r}=\int_0^t f^{(r)}(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds +U\frac{\partial^{r-1}n}{\partial t^{r-1}}, \quad r\leqslant m-1, \tag{B2}$$

$$\frac{\partial^m n}{\partial t^m}=\int_0^t f^{(m)}(t-s)\left[\int_{-\infty}^{\infty}n(s,x+z)\varphi(z)dz-n(s,x)\right]ds +U\frac{\partial^{m-1}n}{\partial t^{m-1}}+\lambda^m\left[\int_{-\infty}^{\infty}n(t,x+z)\varphi(z)dz-n(t,x)\right]. \tag{B3}$$

From Eqs. (B2) and (B3) one can obtain

$$\sum_{r=0}^{m}\lambda^{m-r}\binom{m}{r}\frac{\partial^r n}{\partial t^r}=\lambda^m\int_{-\infty}^{\infty}n(t,x+z)\varphi(z)dz +U\sum_{r=1}^{m}\lambda^{m-r}\binom{m}{r}\frac{\partial^{r-1}n}{\partial t^{r-1}}.$$

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