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Superdiffusion of a random walk driven by an ergodic Markov process with switching

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Abstract
We propose a Markov model with an ergodic two-component switching mechanism that dynamically generates anomalous diffusion. The first component plays the role of a hidden parameter. The second one is the switching component generating the superdiffusion of a random walker and is itself non-Markovian. The model is studied numerically using the Monte Carlo technique.

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1. Introduction

Anomalous diffusion has important applications in many areas in physics, chemistry and life sciences (see, for example, [3, 19, 20] and references therein). It can be characterized by the variance Var $\{Y(t)\}$ of the particle position $Y(t)$ that increases with time as t^γ for large t. The case with $0 < \gamma < 1$ is known as the subdiffusion regime, $\gamma = 1$ corresponds to the normal diffusion, and $\gamma > 1$ is known as the superdiffusion regime. A variety of techniques are available for the description of anomalous transport including the continuous time random walk (CTRW) [24], kinetic equations with fractional space and time derivatives [28, 30] and stochastic differential equations involving non-Markovian random processes [29].

The CTRW model is a standard tool for studying the anomalous diffusion [19, 20]. In CTRW, each step is characterized by a waiting time τ and a displacement (jump) r which are distributed according to the joint probability density function (pdf) $\Phi(\tau, r)$; hence $\psi(\tau) = \int \Phi(\tau, r)\,\mathrm{d}r$ is the pdf of waiting time. In the case of the decoupled density function $\Phi(\tau, r) = \psi(\tau)\lambda(r)$, i.e. when the waiting time and the individual displacement are independent, the subdiffusion regime occurs if the mean waiting time $\bar{\tau} = \int_0^\infty \tau\psi(\tau)\,\mathrm{d}\tau$ is infinite and the symmetrical pdf $\lambda(r)$ has a finite variance $\sigma^2 = \int r^2\lambda(r)\,\mathrm{d}r < \infty$. If the

asymptotic behaviour for the waiting-time density $\psi(\tau)$ for large τ is $\tau^{-1-\zeta}$ with $0 < \zeta < 1$, then the mean waiting time is infinite and the mean-square displacement of the walker is $EY^2(t) \sim \sigma^2 t^\zeta$. When $\bar{\tau}$ is finite, we have the normal diffusion $EY^2(t) \sim Dt$, where $D = \sigma^2/2\bar{\tau}$. Such a decoupling model is inappropriate for a superdiffusion regime [26]. One has to assume that the waiting time and displacement are correlated. For instance, one can introduce the joint pdf as $\Phi(\tau, r) = \delta(\tau - |r|/v)\lambda(r)$. It corresponds to the random walk when the particle moves with a constant speed v; hence the waiting time τ depends on the displacement. When the pdf $\lambda(r)$ has a long tail, i.e. $\lambda(r) \sim r^{-\mu}$ with $2 < \mu < 3$ as $r \to \infty$, this random walk is referred to as a Levy walk [26, 31]. The mean-square displacement for the Levy walk is $EY^2(t) \sim t^{4-\mu}$, i.e. the Levy walk is a superdiffusive process.

An alternative approach to superdiffusion is based on the differential equation for a particle position:

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \mu(t), \tag{1.1}$$

where $\mu(t)$ is a stationary random process with zero mean [29]. From (1.1), one can derive the equation for the mean-square displacement

$$\frac{\mathrm{d}}{\mathrm{d}t} EY^2(t) = 2\int_0^t C_\mu(s)\,\mathrm{d}s, \tag{1.2}$$

where $C_\mu(t)$ is the autocorrelation function for $\mu(t)$: $C_\mu(t) = E\left(\mu(t)\mu(0)\right)$. It follows from (1.2) that the normal diffusion occurs when the autocorrelation function $C_\mu(t)$ decreases quickly enough to make the integral in (1.2) finite as $t \to \infty$. In this case, the asymptotic behaviour of $EY^2(t)$ for large t is $2Dt$, where the diffusion coefficient D is given by the well-known Kubo formula $D = \int_0^\infty C_\mu(s)\,\mathrm{d}s$. Superdiffusion arises when the autocorrelation function $C_\mu(t)$ decays slowly as $t \to \infty$ so that the integral in (1.2) diverges. If this is the case, then the random process $\mu(t)$ is said to have long memory [2]. For the normal diffusion, there exists the characteristic time scale $\vartheta = C_\mu^{-1}(0)\int_0^\infty C_\mu(s)\,\mathrm{d}s$, while for superdiffusion this time is infinite. If the autocorrelation $C_\mu(t)$ tends to zero as the power law $t^{-\zeta}$ when $t \to \infty$, $0 < \zeta < 1$, then the integral diverges as $t \to \infty$ and the mean-square displacement exhibits superdiffusive behaviour $EY^2(t) \sim t^{2-\zeta}$ [3]. Let us note that very often the long-memory correlations $C_\mu(t) \sim t^{-\zeta}$ are introduced into the model (1.1) phenomenologically. An interesting example of a model with a dynamically generated long-memory effect is considered in [3].

Stochastic switching has been of much interest since it is one of the basic processes in many areas of natural sciences, e.g., switching between two metastable states in stochastic resonance theory [10, 18], two-state model for anomalous diffusion [27, 29], two-state gating process for ion channels [11], stochastically driven two-level quantum systems [12], etc. The standard model for switching is a continuous time Markov chain with an exponential distribution of residence times. Its simplest version is the classical telegrapher process which was introduced by Kac [16] and used to analyse the Dirac equation [9]. In this model, the velocity $\mu(t)$ in (1.1) is a dichotomous Markov process taking two values a and $-a$ and switching from one to the other with the constant rate q. The autocorrelation function is exponential: $C_\mu(t) = a^2\exp(-2qt)$, so we have normal diffusion as $t \to \infty$. To model anomalous diffusion, non-Markovian processes have been considered in the recent years. One usually implements the standard renewal theory with arbitrary non-exponential residence time distributions such as stretched exponential $\psi(\tau) \sim \frac{\mathrm{d}}{\mathrm{d}\tau}\exp[-(q\tau)^\alpha]$ or power law $\psi(\tau) \sim \tau^{-\zeta}$ [11].

The main purpose of this paper is to set up a Markov model with a switching mechanism which dynamically generates anomalous diffusion. The developed Markov

switching mechanism induces long-range temporal correlations such that the integral of the corresponding autocorrelation function is divergent as $t \to \infty$. Thus, we do not use given non-exponential residence time distributions; instead we introduce the auxiliary random variable X varying in the interval $(0, 1)$. It obeys the stochastic equation (see section 3)

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X, \mu), \tag{1.3}$$

where μ is a chain taking two values μ_1 and μ_2. The essential feature of this model is that the infinitesimal characteristics q_1 and q_2 of the chain μ depend on the value of the random process X. We note that this dependence makes the switching component $\mu(t)$ non-Markovian. At the same time, the pair (X, μ) defines a Markov process. The particle position $Y(t)$ is determined by the stochastic equation

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(\mu), \tag{1.4}$$

where b takes two values $b_1 = b(\mu_1)$ and $b_2 = b(\mu_2)$, respectively. We note that the triple (X, μ, Y) is a Markov process as well, and therefore the conventional Markovian techniques can be used to study the non-Markovian switching component $\mu(t)$ and the random position $Y(t)$. In particular, we consider the stochastic equation (see section 4)

$$\frac{\mathrm{d}X}{\mathrm{d}s} = \begin{cases} (1 - X)^{1+\alpha}, & \text{if } \mu = \mu_1, \\ -X^{1+\alpha}, & \text{if } \mu = \mu_2, \end{cases} \tag{1.5}$$

with the following infinitesimal characteristics of the chain $\mu(t)$:

$$q_1(x) = (1 - x)^{\beta}, \qquad q_2(x) = x^{\beta}. \tag{1.6}$$

The Markov process (X, μ) is ergodic and has a unique stationary distribution. It follows from (1.5) and (1.6) that the value of the random variable X determines the 'strength' of the states μ_1 and μ_2. In particular, if $X(t)$ is close to zero, then the transition probability is very small and therefore the system is 'trapped' in the regime with $\mu = \mu_2$. This can induce a long-range temporal correlation of $b(\mu)$ and ultimately the anomalous behaviour of the random process $Y(t)$. One of the main results of this paper is that the mean-square displacement $EY^2(t)$ can exhibit the superdiffusive behaviour, namely, $EY^2(t) \sim t^{\gamma}$ as $t \to \infty$ with the exponent $1 < \gamma < 2$.

The switching mechanism suggested in this paper is of interest itself. Let us give an example related to the ion channels in the biological membranes [15]. These channels are stochastically switched between the closed and open states depending on the external conditions. There is experimental evidence that the observed two-state ion channel gating is non-Markovian (see [11] and references therein). The Markov process (X, μ) and its generalizations can be used to analyse the non-Markovian character of ion channel gating. One can think of the random variable X as the concentration of some ligand regulating the two-state gating process. The Markov process (X, μ) with a non-Markovian switching component μ can also be useful for an analysis of the cancer cells of two phenotypes with random switching between cell proliferation and migration [8]. Another possible application of our model is a state-mediated biased random walk for which switching probabilities for a velocity-jump process depend on internal state dynamics [13, 25].

This paper is organized as follows. In section 2, we recall the required facts about chains interacting with ordinary differential equations. In section 3, we consider and analyse a random walk generated by the Markovian switching mechanism. Based on the results of section 3, we propose a concrete model in section 4, for which anomalous diffusion can be observed. This model is studied numerically by the Monte Carlo technique in section 5. A summary of the obtained results is given in section 6.

2. Preliminaries

2.1. Interaction of chains with ordinary differential equations

Consider the system of ordinary differential equations interacting with a chain:

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X, \mu), \tag{2.1}$$

where $X = (X_1, \ldots, X_d)^\top$, $a(x, \mu) = (a_1(x, \mu), \ldots, a_d(x, \mu))^\top$ are d-dimensional vectors and μ is the chain with m states $\mu_1, \ldots, \mu_m$. Let the functions $a_i(x, \mu_j)$, $i = 1, \ldots, d$, $j = 1, \ldots, m$, $x \in \mathbf{R}^d$, satisfy the Lipschitz condition and grow at infinity not faster than a linear function of x. We assume the infinitesimal matrix Q of the chain μ to depend on the state x of X:

$$Q = \begin{bmatrix} -q_1(x) & q_{12}(x) & \cdots & q_{1m}(x) \\ q_{21}(x) & -q_2(x) & \cdots & q_{2m}(x) \\ \vdots & \vdots & . & \vdots \\ q_{m1}(x) & q_{m2}(x) & \cdots & -q_m(x) \end{bmatrix},$$

where $q_i(x)$ and $q_{ij}(x)$ are the continuous non-negative functions bounded in $\mathbf{R}^d$ and the relations

$$\sum_{j \neq i} q_{ij}(x) = q_i(x) \tag{2.2}$$

are fulfilled. Then the system (2.1) generates a Markov process (X, μ) with the infinitesimal generator (see, e.g., [22, 21] and references therein):

$$Ag(x, \mu_i) = \sum_{k=1}^{d} a_k(x, \mu_i) \frac{\partial g}{\partial x_k}(x, \mu_i) - q_i(x) g(x, \mu_i) + \sum_{j \neq i} q_{ij}(x) g(x, \mu_j),$$
$$i = 1, \ldots, m, x \in \mathbf{R}^d. \tag{2.3}$$

Due to (2.3), the following Cauchy problem for the system of hyperbolic equations,

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{d} a_k(x, \mu_i) \frac{\partial u_i}{\partial x_k} + k(x, \mu_i) u_i - q_i(x) u_i + \sum_{j \neq i} q_{ij}(x) u_j + c(x, \mu_i), \tag{2.4}$$

$$u_i(0, x) = f(x, \mu_i), \qquad i = 1, \ldots, m, \tag{2.5}$$

admits a probabilistic representation for its solution. The representation has the form (the Feynman–Kac formula)

$$u_i(t, x) = E f(X_{x,\mu_i}(t), \mu_{x,\mu_i}(t)) \exp\left(\int_0^t k(X_{x,\mu_i}(s), \mu_{x,\mu_i}(s))\, \mathrm{d}s \right)$$
$$+ E \int_0^t \exp\left(\int_0^s b(X_{x,\mu_i}(\vartheta), \mu_{x,\mu_i}(\vartheta))\, \mathrm{d}\vartheta \right) c(X_{x,\mu_i}(s), \mu_{x,\mu_i}(s))\, \mathrm{d}s, \quad t \geqslant 0, \tag{2.6}$$

where $X_{x,\mu_i}(s)$, $\mu_{x,\mu_i}(s)$ is a realization of the Markov process (X, μ) starting from $X(0) = x$, $\mu(0) = \mu_i$. Formula (2.6) is valid, for instance, under the assumption that the functions $f_i(x) = f(x, \mu_i)$, $k_i(x) = k(x, \mu_i)$, $c_i(x) = c(x, \mu_i)$ and the partial derivatives $\partial f_i / \partial x_j$ are continuous bounded functions in $\mathbf{R}^d$. The probabilistic representation (2.6) is useful for

numerics exploiting the Monte Carlo approach (see section 5). In a broader context, see [23] for the use of probabilistic representations in stochastic numerics.

Let an initial distribution of (X, μ) be given by a density $\lambda_i(x)$, i.e.

$$P\{X(0) \in H, \mu(0) = \mu_i\} = \int_H \lambda_i(x)\,\mathrm{d}x, \qquad i = 1, \ldots, m.$$

We assume that $\lambda_i(x)a_k(x, \mu_i)$ have continuous partial derivatives $\partial(\lambda_i a_k)/\partial x_k$ and the integrals

$$I_i = \int_{\mathbf{R}^d} \sum_{k=1}^{d} \frac{\partial(\lambda_i(x)a_k(x, \mu_i))}{\partial x_k}\,\mathrm{d}x, \qquad i = 1, \ldots, m, \tag{2.7}$$

absolutely converge. Then the density $\psi_i(s, x)$ of (X, μ) at time s satisfies the system of differential equations

$$\frac{\partial \psi_i}{\partial s} = -\sum_{k=1}^{d} \frac{\partial(\psi_i a_k(x, \mu_i))}{\partial x_k} - q_i(x)\psi_i + \sum_{j \neq i} q_{ji}(x)\psi_j, \qquad \psi_i(0, x) = \lambda_i(x). \tag{2.8}$$

As a consequence, we obtain that $\lambda_i(x)$ is the stationary distribution, provided the integrals (2.7) absolutely converge, if and only if it satisfies the system

$$-\sum_{k=1}^{d} \frac{\partial(\lambda_i a_k(x, \mu_i))}{\partial x_k} - q_i(x)\lambda_i + \sum_{j \neq i} q_{ji}(x)\lambda_j = 0. \tag{2.9}$$

It should be noted that all what has been said above is correct if one takes some open set $\mathcal{D} \subseteq \mathbf{R}^d$ instead of $\mathbf{R}^d$ provided $X_{x,\mu_i}(t) \in \mathcal{D}$ for any $x \in \mathcal{D}$, $i = 1, \ldots, m$, $t \geqslant 0$. In addition, let us observe that the Markov process (X, μ) is related to the piecewise-deterministic Markov processes considered in [4–6].

2.2. *One-dimensional system with two-state chain*

Let us treat the case when equation (2.1) is scalar ($d = 1$), $X \in (0, 1)$, the chain μ takes two values μ_1 and μ_2 and $a_1(x) := a(x, \mu_1) > 0$, $a_2(x) := a(x, \mu_2) < 0$, $q_1(x) > 0$, $q_2(x) > 0$ in the entire interval $(0, 1)$. Clearly, due to (2.2), we have $q_{12}(x) = q_1(x)$ and $q_{21}(x) = q_2(x)$. In [1], the behaviour of the process $(X(t), \mu(t))$ is studied. The approach of [1] in many respects repeats the well-known investigation of the one-dimensional diffusion on the bounded interval due to Feller (see, for instance, [7, 14]). We are interested in the case when the ends of the interval $(0, 1)$ are unattainable in finite time for the process X (i.e. $\mathcal{D} = (0, 1)$) and in ergodicity of the process. Here we follow [17], where such questions were considered for diffusion equations.

Let τ_{x,μ_i}^{y,μ_j} be the first time when the process $(X_{x,\mu_i}(t), \mu_{x,\mu_i}(t))$ attains (y, μ_j). We recall that the process (X, μ) is called recurrent if for any two points (x, μ_i), (y, μ_j), $x, y \in (0, 1)$, $i, j = 1, 2$,

$$P\left\{\tau_{x,\mu_i}^{y,\mu_j} < \infty\right\} = 1.$$

Denote

$$k_1(x) = \frac{q_1(x)}{a_1(x)}, \qquad k_2(x) = \frac{q_2(x)}{a_2(x)},$$

$$k(x) = k_1(x) + k_2(x), \qquad m(x) = \frac{1}{a_1(x)} - \frac{1}{a_2(x)}.$$

Due to the assumptions made above, we have $k_1(x) > 0$, $k_2(x) < 0$ and $m(x) > 0$ in the entire interval $(0, 1)$.

Introduce the integrals

$$I(0,r] = -\int_0^r k_2(\xi)\exp\left(-\int_\xi^r k(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\xi, \qquad 0<r<1, \tag{2.10}$$

$$I[l,1) = \int_l^1 k_1(\xi)\exp\left(\int_l^\xi k(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\xi, \qquad 0<l<1. \tag{2.11}$$

Note that both boundedness and unboundedness of the integrals (2.10) and (2.11) do not depend on the choice of r and l. It is shown in [1] that if

$$I(0,r] = \infty, \qquad I[l,1) = \infty, \tag{2.12}$$

the ends of the interval $(0,1)$ are unattainable in finite time for the process X. The following theorem is proved in [1].

Theorem 2.1. *The process (X,μ) is recurrent if and only if condition (2.12) is fulfilled.*

It is also proved in [1] that

$$E\tau_{x,\mu_i}^{y,\mu_j} < \infty$$

for any two points (x,μ_i) and (y,μ_j) if and only if

$$K := \int_0^1 m(\xi)\exp\left(-\int_c^\xi k(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\xi < \infty, \qquad 0<c<1. \tag{2.13}$$

Clearly, convergence of this integral does not depend on the choice of c.

The following theorem holds (see [1]).

Theorem 2.2. *The process (X,μ) is ergodic if*

$$I(0,r] = \infty, \qquad I[l,1) = \infty, \qquad K<\infty. \tag{2.14}$$

In the case (2.14), the process (X,μ) has a stationary measure with the density

$$\begin{aligned}
\lambda_1(x) &= \frac{L}{a_1(x)}\exp\left(-\int_c^x k(\zeta)\,\mathrm{d}\zeta\right) = \frac{1}{a_1(x)\cdot\int_0^1 m(\xi)\exp\left(\int_\xi^x k(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\xi},\\
\lambda_2(x) &= -\frac{L}{a_2(x)}\exp\left(-\int_c^x k(\zeta)\,\mathrm{d}\zeta\right) = -\frac{1}{a_2(x)\cdot\int_0^1 m(\xi)\exp\left(\int_\xi^x k(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\xi},
\end{aligned} \tag{2.15}$$

where the constant L is equal to $1/K$ with K from (2.13).

3. Random walk generated by the switching component μ of the Markov process (X,μ)

3.1. Mean and variance of the random walk

Consider the system

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X,\mu), \qquad \frac{\mathrm{d}Y}{\mathrm{d}s} = b(\mu), \tag{3.1}$$

where b takes two values $b_1 = b(\mu_1)$ and $b_2 = b(\mu_2)$, respectively. The system (3.1) defines the Markov process (X,μ,Y), and the component Y is a random walk on the real axis y. Let $X(0)=x$, $Y(0)=y$ and $\mu(0)=\mu_i$. The functions

$$u_i(t,x,y) := EY_{x,y,\mu_i}(t) = y + E\int_0^t b(\mu_{x,\mu_i}(s))\,\mathrm{d}s, \qquad i=1,2,$$

satisfy the system of forms (2.4) and (2.5). From this, we easily derive that the functions

$$u_i(t,x) := u_i(t,x,0) = E\int_0^t b(\mu_{x,\mu_i}(s))\,\mathrm{d}s, \qquad i=1,2$$

satisfy the system

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = a_i(x)\frac{\partial u_i}{\partial x} - q_i(x)u_i + q_i(x)u_j + b(\mu_i), \qquad j\neq i,\\ &u_i(0,x) = 0, \qquad i=1,2. \end{aligned} \tag{3.2}$$

Analogously, considering the solution of the system

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X,\mu), \qquad \frac{\mathrm{d}Y}{\mathrm{d}s} = b(\mu(s)), \qquad \frac{\mathrm{d}Z}{\mathrm{d}s} = 2b(\mu(s))Y,$$

starting from $X(0)=x$, $Y(0)=0$, $Z(0)=0$, $\mu(0)=\mu_i$, it is not difficult to derive that

$$v_i(t,x) := EZ_{x,0,0,\mu_i}(t) = EY^2_{x,0,\mu_i}(t) = E\left[\int_0^t b(\mu_{x,\mu_i}(s))\,\mathrm{d}s\right]^2, \qquad i=1,2, \tag{3.3}$$

satisfy the system

$$\begin{aligned} &\frac{\partial v_i}{\partial t} = a_i(x)\frac{\partial v_i}{\partial x} - q_i(x)v_i + q_i(x)v_j + 2b(\mu_i)u_i, \qquad j\neq i,\\ &v_i(0,x) = 0, \qquad i=1,2. \end{aligned} \tag{3.4}$$

Let the initial data $X(0)$, $\mu(0)$ be random and let the initial distribution of (X,μ) be given by a density $\lambda_i(x)$. Introduce the functions

$$u(t) = EE\left[\int_0^t b(\mu_{X(0),\mu(0)}(s))\,\mathrm{d}s|X(0),\mu(0)\right] = \int_0^1 (u_1(t,x)\lambda_1(x) + u_2(t,x)\lambda_2(x))\,\mathrm{d}x, \tag{3.5}$$

$$v(t) = EE\left[\left(\int_0^t b(\mu_{X(0),\mu(0)})\,\mathrm{d}s\right)^2 |X(0),\mu(0)\right] = \int_0^1 (v_1(t,x)\lambda_1(x) + v_2(t,x)\lambda_2(x))\,\mathrm{d}x. \tag{3.6}$$

Now suppose that the process (X,μ) is ergodic and the density $\lambda_i(x)$ is stationary. Then (see (3.2)),

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \int_0^1 \left(\frac{\partial u_1}{\partial t}\lambda_1(x) + \frac{\partial u_2}{\partial t}\lambda_2(x)\right)\mathrm{d}x\\ &= \int_0^1 \left[a_1(x)\frac{\partial u_1}{\partial x} - q_1(x)u_1 + q_1(x)u_2 + b(\mu_1)\right]\lambda_1(x)\,\mathrm{d}x\\ &\quad + \int_0^1 \left[a_2(x)\frac{\partial u_2}{\partial x} - q_2(x)u_2 + q_2(x)u_1 + b(\mu_2)\right]\lambda_2(x)\,\mathrm{d}x\\ &= a_1(x)\lambda_1(x)u_1|_0^1 + \int_0^1 \left[-\frac{\partial(a_1\lambda_1)}{\partial x}u_1 - q_1\lambda_1u_1 + q_1\lambda_1u_2 + b(\mu_1)\lambda_1\right]\mathrm{d}x\\ &\quad + a_2(x)\lambda_2(x)u_2|_0^1 + \int_0^1 \left[-\frac{\partial(a_2\lambda_2)}{\partial x}u_2 - q_2\lambda_2u_2 + q_2\lambda_2u_1 + b(\mu_2)\lambda_2\right]\mathrm{d}x. \end{aligned} \tag{3.7}$$

If $a_i(0)\lambda_i(0) = a_i(1)\lambda_i(1) = 0, i = 0, 1$ (this is a rather natural assumption, see, e.g., the model in the following section), the substitutions in (3.7) vanish. Because the density $\lambda_i(x)$ satisfies (2.9), we get

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \int_0^1 (b(\mu_1)\lambda_1(x) + b(\mu_2)\lambda_2(x))\,\mathrm{d}x. \tag{3.8}$$

One can assume that

$$\int_0^1 (b(\mu_1)\lambda_1(x) + b(\mu_2)\lambda_2(x))\,\mathrm{d}x = 0 \tag{3.9}$$

without loss of generality for further analysis. Due to this assumption, $v(t)$ is the variance of the random walk $Y(t)$ when the process (X, μ) starts from the stationary distribution.

Analogously, we can derive

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 2\int_0^1 (b(\mu_1)u_1(t, x)\lambda_1(x) + b(\mu_2)u_2(t, x)\lambda_2(x))\,\mathrm{d}x. \tag{3.10}$$

The right-hand side of (3.9) can be transformed as

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= 2\int_0^1 (b(\mu_1)E\int_0^t b(\mu_{x,\mu_1}(s))\,\mathrm{d}s\cdot\lambda_1(x) + b(\mu_2)E\int_0^t b(\mu_{x,\mu_2}(s))\,\mathrm{d}s\cdot\lambda_2(x))\,\mathrm{d}x \\ &= 2\int_0^t Eb(\mu(0))b(\mu(s))\,\mathrm{d}s = 2\int_0^t r(s)\,\mathrm{d}s,\end{aligned}$$

where for any $\vartheta \geqslant 0$

$$r(s) = Eb(\mu(0))b(\mu(s)) = Eb(\mu(\vartheta))b(\mu(\vartheta + s))$$

is the covariance function of the component μ of the stationary process (X, μ) starting from the distribution defined by the density $\lambda_i(x)$.

3.2. Normal and anomalous diffusion

If μ does not depend on X, i.e. q_1, q_2 are positive constants (μ is a classic Markov chain in this case), the variance $v(t)$ can be easily evaluated (for simplicity, we take $q_1 = q_2 = q$, $b(\mu_1) = 1$, $b(\mu_2) = -1$):

$$v(t) = \frac{1}{q}t - \frac{1}{2q^2}(1 - \mathrm{e}^{-2qt}) \tag{3.11}$$

and

$$\lim_{t\to\infty}\frac{\ln v(t)}{\ln t} = 1. \tag{3.12}$$

Equality (3.12) is fulfilled for many random walks and, in particular, for the Brownian motion. One usually calls such random walks the normal diffusion.

In general, we have

$$0 < \liminf_{t\to\infty}\frac{\ln v(t)}{\ln t} \leqslant \limsup_{t\to\infty}\frac{\ln v(t)}{\ln t} \leqslant 2.$$

If

$$\lim_{t\to\infty}\frac{\ln v(t)}{\ln t} = 2,$$

one says that ballistic motion takes place. One says that we have anomalous diffusion if

$$1 < \liminf_{t\to\infty}\frac{\ln v(t)}{\ln t} \leqslant \limsup_{t\to\infty}\frac{\ln v(t)}{\ln t} < 2$$

(superdiffusion), or if

$$0 < \lim\inf_{t\to\infty} \frac{\ln v(t)}{\ln t} \leqslant \lim\sup_{t\to\infty} \frac{\ln v(t)}{\ln t} < 1 \quad \text{(subdiffusion)}.$$

If there exist the limits

$$\gamma_{x,\mu_i} = \lim_{t\to\infty} \frac{\ln v_i(t,x)}{\ln t} = \lim_{t\to\infty} \frac{\ln E\big[\int_0^t b(\mu_{x,\mu_i}(s))\,\mathrm{d}s\big]^2}{\ln t},$$

$$\gamma = \lim_{t\to\infty} \frac{\ln v(t)}{\ln t} = \lim_{t\to\infty} \frac{\ln E[E\big[\int_0^t b(\mu_{X(0),\mu(0)})\,\mathrm{d}s\big]^2 | X(0),\mu(0)]}{\ln t},$$

then the variance of Y at large time t is approximately proportional to $t^{\gamma_{x,\mu_i}}$ or t^{γ}, respectively. Apparently, as a rule, $\gamma_{x,\mu_i} = \gamma$ for any x, μ_i.

Although we have undertaken a number of serious attempts to find the exponents γ and γ_{x,μ_i} analytically, we could not obtain any sufficiently complete results. We believe that this problem is both difficult and interesting from the analytical point of view. In this paper, we restrict ourselves to the numerical investigation of the problem. For calculating $v_i(t,x)$ and $v(t,x)$, one can use both the probabilistic approach and the deterministic one. The deterministic approach consists in solving the system (3.4). Another deterministic approach rests on the Laplace transform. Here we use the Monte Carlo technique which is very simple and gives accurate results in calculating the function

$$\gamma(t) := \frac{\ln v(t)}{\ln t}, \qquad 0 \leqslant t \leqslant T,$$

for sufficiently large T. The results allow us to derive correct (from the practical point of view) conclusions for infinite time.

4. Model

In this section, we give a concrete realization of the Markovian switching mechanism proposed in the previous section. Consider the scalar equation in the interval $(0,1)$

$$\frac{\mathrm{d}X}{\mathrm{d}s} = a(X,\mu) = \begin{cases} a_1(X) = (1-X)^{1+\alpha}, & \text{if } \mu = \mu_1, \\ a_2(X) = -X^{1+\alpha}, & \text{if } \mu = \mu_2, \end{cases} \tag{4.1}$$

with the infinitesimal characteristics of the chain μ:

$$q_1(x) = (1-x)^{\beta}, \qquad q_2(x) = x^{\beta}.$$

We have

$$k_1(x) = (1-x)^{\beta-1-\alpha}, \qquad k_2(x) = -x^{\beta-1-\alpha},$$

$$k(x) = (1-x)^{\beta-1-\alpha} - x^{\beta-1-\alpha}, \qquad m(x) = \frac{1}{(1-x)^{1+\alpha}} + \frac{1}{x^{1+\alpha}},$$

and

$$I(0,r] = \exp\left(-\frac{1}{\alpha-\beta}((1-r)^{\beta-\alpha} + r^{\beta-\alpha})\right) \times \int_0^r \exp\left(\frac{1}{\alpha-\beta}((1-\xi)^{\beta-\alpha} + \xi^{\beta-\alpha})\right) \xi^{\beta-1-\alpha}\,\mathrm{d}\xi, \qquad \beta \neq \alpha,$$

$$I(0,r] = r(1-r)\int_0^r \frac{\mathrm{d}\xi}{\xi^2(1-\xi)}, \qquad \beta = \alpha.$$

We see from here that $I(0,r] = \infty$ iff $\beta \leqslant \alpha$. Analogously, $I[l,1) = \infty$ iff $\beta \leqslant \alpha$. In this case, the process (X,μ) is recurrent (see theorem 2.1).

Further,

$$K = \exp\left\{\frac{1}{\alpha-\beta}((1-c)^{\beta-\alpha}+c^{\beta-\alpha})\right\}\int_0^1\left(\frac{1}{(1-\xi)^{1+\alpha}}+\frac{1}{\xi^{1+\alpha}}\right)$$
$$\times\exp\left\{-\frac{1}{\alpha-\beta}((1-\xi)^{\beta-\alpha}+\xi^{\beta-\alpha})\right\}\mathrm{d}\xi, \qquad \beta\neq\alpha,$$
$$K = \frac{1}{c(1-c)}\int_0^1\left(\frac{\xi}{(1-\xi)^{\alpha}}+\frac{1-\xi}{\xi^{\alpha}}\right)\mathrm{d}\xi, \qquad \beta=\alpha,$$

and we see that the conditions of theorem 2.2 are fulfilled iff $\beta<\alpha$ or $\beta=\alpha<1$. So, we get the following proposition.

Proposition 4.1. *If $\beta<\alpha$ or $\beta=\alpha<1$, then the process (X,μ) generated by equation (4.1) is ergodic and its stationary distribution is defined by the density $\lambda_i(x)$, where in the case $\beta<\alpha$*

$$\lambda_1(x) = \frac{C}{(1-x)^{\alpha+1}}\exp\left\{-\frac{1}{\alpha-\beta}((1-x)^{\beta-\alpha}+x^{\beta-\alpha})\right\},$$
$$\lambda_2(x) = \frac{C}{x^{\alpha+1}}\exp\left\{-\frac{1}{\alpha-\beta}((1-x)^{\beta-\alpha}+x^{\beta-\alpha})\right\},$$
$$C = \left[\int_0^1\left(\frac{1}{(1-\xi)^{\alpha+1}}+\frac{1}{\xi^{\alpha+1}}\right)\exp\left\{-\frac{1}{\alpha-\beta}((1-\xi)^{\beta-\alpha}+\xi^{\beta-\alpha})\right\}\mathrm{d}\xi\right]^{-1},$$

and in the case $\beta=\alpha<1$

$$\lambda_1(x) = \frac{(1-\alpha)(2-\alpha)}{2}\frac{x}{(1-x)^{\alpha}}, \qquad \lambda_2(x) = \frac{(1-\alpha)(2-\alpha)}{2}\frac{1-x}{x^{\alpha}}.$$

We note that here we give the simple and natural realization of the switching mechanism from section 3. To reflect specific features of a particular physical system, one can adjust the choice of a and of the characteristics of the chain μ.

5. Numerical analysis

In this section, we study the model (3.1) and (4.1) numerically. For definiteness, in (3.1) we take

$$b(\mu) = \begin{cases} -1, & \text{if } \mu=\mu_1,\\ 1, & \text{if } \mu=\mu_2. \end{cases} \tag{5.1}$$

We first briefly describe the numerical algorithm used in this study and then present the numerical results.

We approximate the Markov process (X,μ,Y) from (3.1), (4.1) and (5.1) with $X(0)=x$, $\mu(0)=\mu_0$ and $Y(0)=0$, respectively, by the chain $(\bar X,\bar\mu,\bar Y)$ as follows. We choose a fixed time step $h>0$. We start the chain from the given values: $\bar X(0)=x$, $\bar\mu(0)=\mu_0$, $\bar Y(0)=0$ and assign $\theta_0=0$. We simulate the random time τ_1 according to the exponential distribution with the parameter $\lambda=q_{\bar\mu(\theta_0)}(\bar X(\theta_0))$. If $\tau_1>h$, we put the time increment $\Delta\theta_1=h$ and $\bar\mu(\theta_1)=\bar\mu(\theta_0)$, $\theta_1=\theta_0+\Delta\theta_1$; otherwise $\Delta\theta_1=\tau_1$ and $\bar\mu(\theta_1)$ is assigned the other state than $\bar\mu(\theta_0)$. We exactly integrate the system (3.1), (4.1), (5.1) in the interval $[0,\theta_1]$ with the initial data $\bar X(\theta_0)=x$, $\bar\mu(\theta_0)=\mu_0$, $\bar Y(\theta_0)=0$, respectively, and thus obtain $\bar X(\theta_1)$, $\bar Y(\theta_1)$. Starting from $\bar X(\theta_1)$, $\bar\mu(\theta_1)$, $\bar Y(\theta_1)$ and doing analogously to the simulation at the first step, we obtain $\bar X(\theta_2)$, $\bar\mu(\theta_2)$, $\bar Y(\theta_2)$, respectively, and so on. In order to simulate characteristics

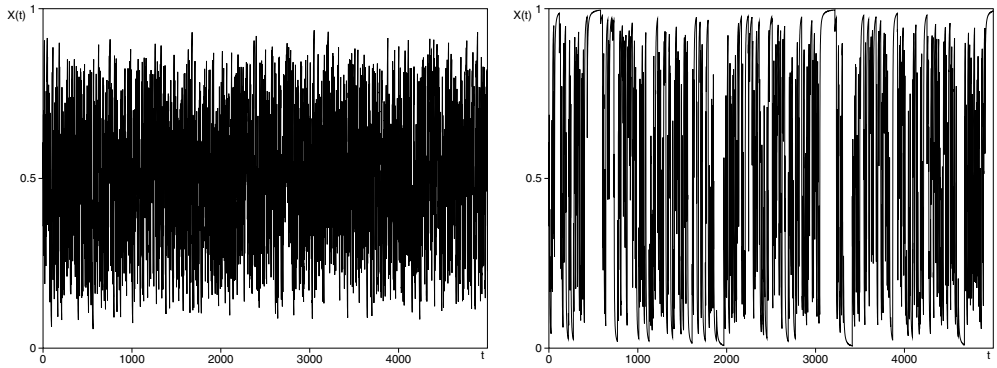

Figure 1. A sample trajectory of $X(t)$ for $\alpha = 0.9$ and $\beta = 0.2$ (left) and $\beta = 0.8$ (right). The simulation is done with $h = 0.01$, $X(0) = 0.5$, $Y(0) = 0$ and $\mu(0) = \mu_1$.

of the random walk, e.g. $\mathrm{Var}(Y)$, on the time interval $[0, T]$, we perform M independent Monte Carlo runs. During each Monte Carlo run $m = 1, \ldots, M$, we get the sequence $\theta_l^{(m)}$, $\bar{X}^{(m)}(\theta_l^{(m)})$, $\bar{\mu}^{(m)}(\theta_l^{(m)})$, $\bar{Y}^{(m)}(\theta_l^{(m)})$, $l = 0, \ldots, \varkappa^{(m)}$, with $\theta_{\varkappa^{(m)}}^{(m)}$ being the first moment that is larger than T. Then we put $\bar{X}^{(m)}(t_k)$, $\bar{\mu}^{(m)}(t_k)$, $\bar{Y}^{(m)}(t_k)$, $k = 0, \ldots, N$, $N = T/h$, equal to $\bar{X}^{(m)}(\theta_l^{(m)})$, $\bar{\mu}^{(m)}(\theta_l^{(m)})$, $\bar{Y}^{(m)}(\theta_l^{(m)})$, respectively, with $\theta_l^{(m)}$ being the closest time moment to t_k. When (x, μ_0) is distributed with a given density $\lambda_i(x)$, $i = 1, 2$, we start the Monte Carlo procedure by simulating $\mu_0^{(m)}$ equal to μ_1 or μ_2 with probability $1/2$ and then by drawing $x^{(m)}$ according to the density $\lambda_{\mu_0^{(m)}}(x)$. It is quite clear that for sufficiently small h the obtained $\bar{X}(t_k)$, $\bar{\mu}(t_k)$, $\bar{Y}(t_k)$ are close to the exact values $X(t_k)$, $\mu(t_k)$, $Y(t_k)$, respectively. We calculate the variance $\mathrm{Var}(Y_{x,0,\mu_0}(T))$ as

$$\mathrm{Var}(Y(T)) \approx \frac{1}{M}\sum_{m=1}^{M}\left[\bar{Y}^{(m)}_{x^{(m)},0,\mu_0^{(m)}}(T)\right]^2 - \left[\frac{1}{M}\sum_{m=1}^{M}\bar{Y}^{(m)}_{x^{(m)},0,\mu_0^{(m)}}(T)\right]^2.$$

Now we present some results of the numerical experiments. We take α, β from the range $0 < \beta < \alpha$ which ensures ergodicity of (X, μ) (see proposition 4.1). We note that the case $\alpha = \beta < 1$ (which is on the boundary of the set of values (α, β) for which the chain (X, μ) is ergodic) is difficult from the numerical point of view, and it is not considered here.

Figure 1 gives typical behaviour of $X(t)$ for different values of α and β while figure 2 illustrates typical behaviour of $Y(t)$. When γ is essentially larger than 1 as in the case $\alpha = 0.9$, $\beta = 0.8$ (cf table 1), the trajectory $Y(t)$ has a clear Levy-flight effect (see occasional very large steps in figure 2(right)). In the case $\alpha = 0.9$, $\beta = 0.2$ the exponent γ is close to 1 and the walk has behaviour very close to the normal diffusion (see figure 2 (left)). We also see in figure 1 that $X(t)$ has more persistency in the case of $\alpha = 0.9$, $\beta = 0.8$ than in the case $\alpha = 0.9$, $\beta = 0.2$.

Figure 3 demonstrates the behaviour of $\gamma(t)$

$$\gamma(t) \approx \bar{\gamma}(t) = \frac{\ln \mathrm{Var}\bar{Y}(t)}{\ln t},$$

for various α and β on a long time interval, while table 1 gives the values of $\bar{\gamma}(T)$ for $T = 10^4$. We see that for a long period of time, the random walker $Y(t)$ has the superdiffusion regime.

It follows from the experiments (see table 1) that for a fixed α the exponent $\gamma(T)$ decreases with the decrease of β, while for a fixed β the exponent $\gamma(T)$ slightly decreases

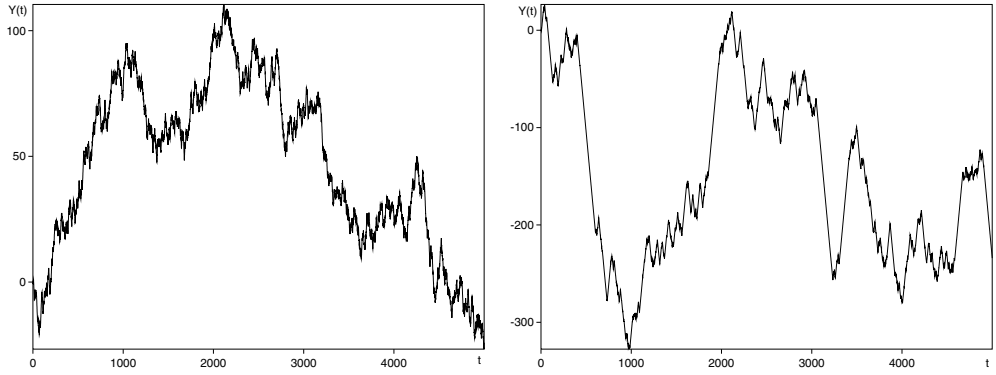

Figure 2. A sample trajectory of $Y(t)$ for $\alpha = 0.9$ and $\beta = 0.2$ (left) and $\beta = 0.8$ (right). The other parameters are the same as in figure 1.

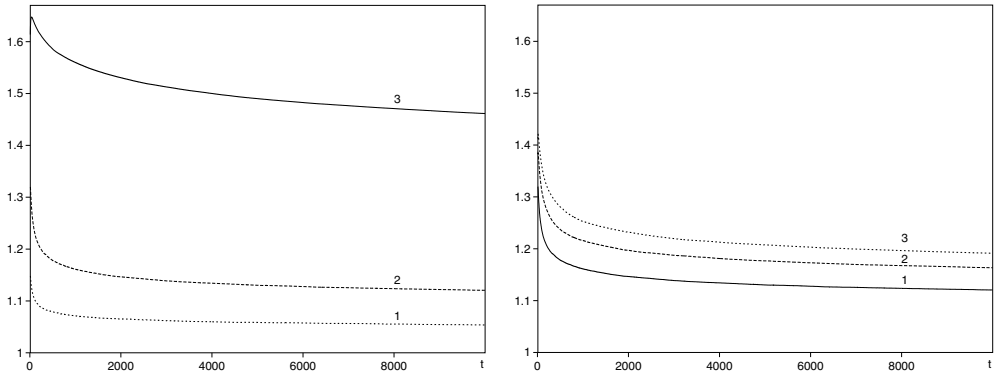

Figure 3. Behaviour of $\bar{\gamma}(t)$. Left: $\alpha = 0.9$ and $\beta = 0.2$ (curve 1), $\beta = 0.4$ (curve 2), $\beta = 0.8$ (curve 3). Right: $\beta = 0.4$ and $\alpha = 0.9$ (curve 1), $\alpha = 0.6$ (curve 2), $\alpha = 0.5$ (curve 3). The other parameters are the same as in table 1.

Table 1. The values of $\bar{\gamma}(T) = \ln \mathrm{Var}(\bar{Y}_{x,0,\mu_0}(T))/\ln T$ for $T = 10^4$, $h = 0.02$, $M = 10^5$ and for various α and β; the initial data (x, μ_0) are distributed according to the stationary density $\lambda_i(x)$. Here the Monte Carlo error of $\mathrm{Var}(\bar{Y}_{x,0,\mu_0})$ is not larger than 1.5%.

	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 1.5$	$\alpha = 3$
$\beta = 0.2$	1.08	1.07	1.07	1.05	1.04	1.03
$\beta = 0.4$	–	1.19	1.16	1.12	1.09	1.06
$\beta = 0.8$	–	–	–	1.46	1.21	1.12
$\beta = 1$	–	–	–	–	1.31	1.16
$\beta = 2$	–	–	–	–	–	1.85

with the increase of α. With β tending to 0, the behaviour becomes closer and closer to the normal diffusion, which is natural since for $\beta = 0$ the infinitesimal characteristics of the chain μ do not depend on x and μ is a classic Markov chain in this case (see (3.10) and (3.11)). For a fixed β and α tending to ∞, the behaviour also becomes closer and closer to the normal diffusion.

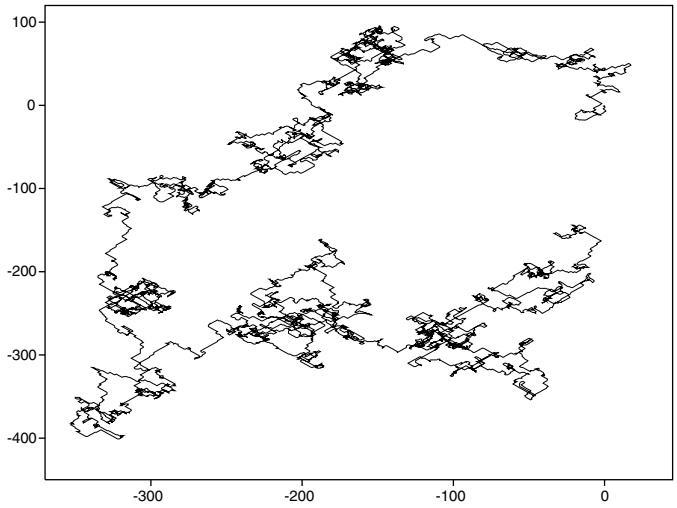

Figure 4. A sample phase trajectory $(Y_1(t), Y_2(t))$ starting from $(0, 0)$ at $t = 0$. The simulation is done for $\alpha = 0.5$, $\beta = 0.4$ and $0 \leqslant t \leqslant 10\,000$ and with $h = 0.01$

Figure 4 gives a sample phase trajectory $(Y_1(t), Y_2(t))$, where $Y_i(t)$ are independent and modelled by (3.1), (4.1) and (5.1). The presented trajectory corresponds to $\alpha = 0.5$ and $\beta = 0.4$, when $\bar{\gamma}(t) \approx 1.19$ (see table 1). One can observe the typical superdiffusion behaviour: there are long, rare steps which mostly determine the overall position of the particle.

6. Conclusions

In this paper, we develop the Markov model with a switching mechanism that dynamically generates the superdiffusive behaviour of a random walker. The model involves three components: the 'hidden' variable X, the non-Markovian switching component μ and the random walker position Y. The main feature of this model is that the transition probabilities of the switching component μ depend on the variable X. This dependence can induce a long-temporal correlation of the component μ and ultimately the superdiffusion of the random walker. We obtain the conditions under which the Markov process (X, μ) is ergodic. We also find explicit expressions for the corresponding stationary distributions. By using the Monte Carlo technique, we demonstrate that the variance of Y is proportional to t^{γ} with $\gamma > 1$ for large t.

The model can be generalized in many ways, e.g., by modelling the 'hidden' variable X via a stochastic differential equation instead of the ordinary one used here. We believe that the proposed Markov model with a non-Markovian switching component can be useful for a variety of physical and biological applications including the gating process for ion channels, migration and proliferation dichotomy in tumour cell invasion, stochastic resonance theory, anomalous diffusion, etc.

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