Analysis of fronts in reaction-dispersal processes

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The existence of traveling wave front solutions with a minimum speed selected for reaction-dispersal processes is studied. We obtain a general existence condition in terms of the waiting time and dispersal distance probability distribution functions and we detail this result for situations of ecological interest. In particular, when particles disperse according to jumps of short length and any waiting time probability distribution function, we show that the minimum speed selection for traveling wave fronts is not always possible, so the waiting time and the dispersal distance distributions cannot be arbitrarily chosen.

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I. INTRODUCTION

It is well known that when diffusive processes couple to reaction, front solutions exist [1]. When the reaction term is of logistic type then the emerging fronts (pulled fronts) from sufficiently localized initial conditions travel with minimum possible speed. The simplest case is the parabolic reaction-diffusion equation also called Fisher's equation where particles jump according to diffusion and wait a short time between successive jumps. Some extensions of the Fisher's equation have been made. For example, if waiting times are not so small but particles disperse by diffusion one has the hyperbolic reaction-diffusion equation [2]. In ecological modeling it is interesting to consider long range dispersals but short waiting times; then one has to deal with integrodifferential equations [3]. The microscopic properties of the underlying transport may be described in general by a waiting time $\varphi(t)$ and a dispersal distance $\Phi(x)$ probability distribution function (PDF). The first one gives the probability that a particle waits a time t between successive jumps and the second one gives the probability of performing a jump of distance x.

By means of the Hamilton-Jacobi method [4] we study the minimum speed selection of pulled fronts and we show that these fronts exist only when both the Laplace transform of $\varphi(t)$ and the bilateral transform of $\Phi(x)$ fulfill a certain restriction. Then, traveling wave front solutions with a minimum speed selected do not exist for arbitrary distributions, an interesting result for the modelization of real systems which is here illustrated for some limiting situations. First, we consider that particles disperse by diffusion but wait a time between jumps according to a general waiting time PDF and for two specific cases of ecological interest: the Poisson (which is usual when random processes are involved) and the Dirac delta. The first one is in fact a family of PDF's with a long tail; only for some of them traveling wave front solutions with a minimum speed selected may exist, as we shall see. The second one models a single waiting time which can be arbitrarily large. In this case we show that the existence condition fails unless one considers small waiting times.

The second limiting situation considers that particles, waiting a short time between jumps, may disperse according to a general dispersal distance distribution. We show that traveling wave front solutions with a minimum speed selected exist in this case for any dispersal kernel. Moreover, the specific case of a Laplace (leptokurtic) dispersal distribution is studied, as it has major interest on ecological applications (see, for example, Ref. [5]).

II. REACTION-DISPERSAL MODELS

We derive the evolution equation for the reaction-dispersal process according to the continuous-time random walk theory (CTRW). The quantity which defines the motion is the probability distribution function (PDF) $\Psi(x,t)$ of the particle performing a jump of length x after waiting a time t at its starting point. If $P(x,t)$ is the probability density of arriving at point x at time t and $\rho(x,t)$ is the probability density of being at point x at time t, we have

$$P(x,t) = \int_{\mathbb{R}} dx' \int_0^t dt' \Psi(x-x',t-t')P(x',t') + P(x,t=0)\delta(t) + g(x,t),$$

$$\rho(x,t) = \int_0^t dt' \phi(t-t')P(x,t'), \tag{1}$$

where $\phi(t)$ is the probability of remaining at least a time t on the point before proceeding with another jump. If $\varphi(t) = \int dx \Psi(x,t)$ is defined as the waiting time PDF, in the Fourier-Laplace space, Eqs. (1) can be rewritten in the closed form [6],

$$\rho(x,t) = \int_0^t dt' \varphi(t') \int_{\mathbb{R}} dx' \Phi(x')\rho(x-x',t-t') + \int_0^t dt' \phi(t') f(\rho(x,t-t')). \tag{2}$$

In this expression, f is the local growth function and it depends explicitly on ρ as a nonlinear function. We will consider that it is of the Fisher-Kolmogorov-Petrovski-Piskunov type [1,7] $f=r\rho F(\rho)$ where r is the constant growth rate and $F(\rho)$ is such that $F(0)=1$. Making use of the Hamilton-Jacobi techniques we have shown before [8] that in this case Eq. (2) becomes the Hamilton-Jacobi equation

$$\frac{1}{\hat{\varphi}(H)} = \hat{\Phi}(p) + \frac{r}{H}\left(\frac{1}{\hat{\varphi}(H)} - 1\right), \tag{3}$$

with the hatted distributions defined as the transforms of $\varphi(t)$ and $\Phi(x)$,

$$\hat{\varphi}(H) = \int_0^{\infty} e^{-Ht}\varphi(t)dt, \quad \hat{\Phi}(p) = \int_{-\infty}^{\infty} e^{px}\Phi(x)dx, \tag{4}$$

and the speed of the front given by the expressions

$$v = \frac{H}{p}, \quad \frac{dH}{dp} = \frac{H}{p}. \tag{5}$$

The latter is nothing but the existence condition of a minimum speed if $d^2H/dp^2 > 0$. So, joining both equations in Eq. (5) the expression for v can be written as

$$v = \min_H \frac{H}{p(H)} \text{ or } v = \min_p \frac{H(p)}{p}. \tag{6}$$

Now, one may wonder if there exists any general condition for the existence of a minimum in these expressions. This condition could be of great theoretical and practical interest, as it would determine the form of the Laplace and bilateral transform for the distributions of waiting times $\hat{\varphi}(H)$ and dispersal distances $\hat{\Phi}(p)$.

The existence of a minimum in Eq. (6) requires that

$$\lim_{H\to 0} \frac{d\left(\frac{H}{p(H)}\right)}{dH} < 0 \text{ and } \lim_{H\to\infty} \frac{d\left(\frac{H}{p(H)}\right)}{dH} \geqslant 0 \tag{7}$$

or

$$\lim_{p\to 0} \frac{d\left(\frac{H(p)}{p}\right)}{dp} < 0 \text{ and } \lim_{p\to\infty} \frac{d\left(\frac{H(p)}{p}\right)}{dp} \geqslant 0. \tag{8}$$

First of all, we need to show that Eqs. (7) and (8) are equivalent. An easy way to do this is by proving that $dH/dp > 0$ [as $dv/dp = (dH/dp)(dv/dH)$]. Then, we must study the expression

$$\frac{\partial H}{\partial p} = \frac{\hat{\Phi}'(p)}{-\frac{\hat{\varphi}'}{\hat{\varphi}^2}\left(1 - \frac{a}{H\tau}\right) + \frac{r}{H^2}\left(\frac{1-\hat{\varphi}}{\hat{\varphi}}\right)}, \tag{9}$$

where we define $\hat{\Phi}'(p) = d\hat{\Phi}(p)/dp$ and $\hat{\varphi}'(H) = d\hat{\varphi}(H)/dH$. The numerator and the denominator in the expression above can be proved to be positive separately. For the numerator, we find

$$\hat{\Phi}'(p) = \int_{-\infty}^{\infty} xe^{px}\Phi(x)dx = \int_0^{\infty} x(e^{px} - e^{-px})\Phi(x)dx > 0, \tag{10}$$

where we have used the isotropy of the kernel, i.e., $\Phi(-x) = \Phi(x)$.

For the expression in the denominator, we start using the normalization of $\Phi(x)$ and $\varphi(t)$ to obtain

$$\begin{aligned}\hat{\Phi}(p) &= \int_{-\infty}^{\infty} e^{px}\Phi(x)dx \\ &= \int_0^{\infty} (e^{px} + e^{-px})\Phi(x)dx \\ &> \inf_{x\in[0,\infty)} (e^{-px} + e^{px}) \int_0^{\infty} \Phi(x)dx = 1,\end{aligned} \tag{11}$$

$$\hat{\varphi}(H) = \int_0^{\infty} e^{-Ht}\varphi(t)dt < \sup_{t\in[0,\infty)} (e^{-Ht}) \int_0^{\infty} \varphi(t)dt = 1, \tag{12}$$

$$\hat{\varphi}'(H) = -\int_0^{\infty} te^{-Ht}\varphi(t)dt < 0. \tag{13}$$

Finally, from the results (11) and (12), Eq. (3) lead us to

$$\left(1 - \frac{a}{H\tau}\right) = \left(\frac{\hat{\varphi}}{1-\hat{\varphi}}\right)\frac{1}{\hat{\Phi}(p) - 1} > 0. \tag{14}$$

So, we have proved the condition $\partial H/\partial p > 0$. Now we know that Eqs. (7) or (8) can be analyzed indistinctly. Hence we just need to prove what are the necessary conditions for Eq. (7).

(i) Limit $H\tau \to 0 (p \to 0)$: From Eq. (14), it can be seen that the solution is restricted for values of $H\tau$ higher than a. Then, this limit is equivalent to $H\tau \to a^+$. The derivative in Eq. (7) reads

$$\begin{aligned}\frac{d\left(\frac{H}{p(H)}\right)}{dH} &= \frac{1}{2}\sqrt{\frac{1-\hat{\varphi}(H)}{\hat{\varphi}(H)}}\frac{\hat{\varphi}'(H)}{[1-\hat{\varphi}(H)]^2}\frac{(H\tau)^{3/2}}{\sqrt{H\tau - a}} \\ &\quad + \frac{1}{2}\tau\sqrt{\frac{\hat{\varphi}(H)}{1-\hat{\varphi}(H)}}\frac{\sqrt{H\tau}(2H\tau - 3a)}{(H\tau - a)^{3/2}}.\end{aligned} \tag{15}$$

Taking into account the expressions (12) and (13) in Eq. (15) we find that the condition

$$\lim_{H\to a^+} \frac{d\left(\frac{H}{p(H)}\right)}{dH} < 0 \tag{16}$$

holds for any value of $H\tau$.

(ii) Limit $H\tau \to \infty (p \to \infty)$: This limit cannot be analyzed straight, so we first need to expand by Taylor up to order M in $\hat{\varphi}(H)^{-1}$ and up to order N in $\hat{\Phi}(p)$,

$$\hat{\varphi}(H)^{-1} = \sum_{m=0}^{M} \beta_m H^m,$$

$$\hat{\Phi}(p) = \sum_{n=0}^{N} \alpha_n p^n, \tag{17}$$

where the coefficients α_i and β_i depend on the generating moments of $\Phi(x)$ and $\varphi(t)$, respectively. In the limit $H\tau \to \infty (p \to \infty)$ the expansions in Eq. (17) are dominated by the term with the highest exponent and now the Hamilton-Jacobi equation (3) leads us to the relation

$$\alpha_N p^N \simeq \beta_M H^M. \tag{18}$$

That is, $p \sim H^{M/N}$ and we can write $v = H(p)/p = cH^{1-M/N}$ with c a positive constant. This allows us to affirm that the second condition in Eq. (7) only holds for $M \leqslant N$.

As a result, we have found that traveling wave solutions with a minimum speed selected are only possible if the highest order in the truncation of the Taylor expansion of the inverse of $\hat{\varphi}(H)$ is not higher than the order in the truncation of the Taylor expansion of $\hat{\Phi}(p)$. In previous works [2,9], it has been shown that the meaning of cutting the Taylor expansion in $\hat{\Phi}(p)$ at a low order is that our measurements are then restricted to much further distances than the characteristic distance of dispersal α (that is $|x| \gg \alpha$), that is, we are in the asymptotic regime (or, analogously, one may say that the distances of dispersal are short). Likewise, expanding $\hat{\varphi}(H)$ [or $\hat{\varphi}(H)^{-1}$] just up to a certain order is equivalent to assume that our observations are only valid for much greater times than the characteristic waiting time τ (that is $t \gg \tau$) which is also equivalent to consider that waiting times are short.

From these arguments, the condition for the existence of v found above is easy to interpret: if the distances where the fronts can be analyzed are very high (N is low), then the times of our measurements must be correspondingly high (M not higher than N) and it is not possible to determine what happens at the initial stages. On the contrary, when we are restricted to high (asymptotic) times (N is low), we have that there are no restrictions on the distances we may observe; as the front has already formed, we can study both the short and the long distances.

Besides the intuitive interpretation given here, we want to stress that the condition $M \leqslant N$ we have found is of great practical interest, since it shows that we cannot choose arbitrarily the distributions $\varphi(t)$ and $\Phi(x)$ if we want to guarantee the existence of traveling wave front solutions with a minimum speed selected of reaction-dispersal equations; it is an important fact that experimentalists should take into account for the modelization of real systems. In the next sections, some specific examples are reported in order to illustrate the above results.

III. LONG WAITING TIMES AND SHORT DISTANCE OF DISPERSAL

We assume in this section that the dispersal distance PDF describes jumps of short length. Then, $\hat{\Phi}(p)$ may be expanded in Taylor series up to second order to yield $\hat{\Phi}(p) \simeq 1 + a_1 p + a_2 p^2$ with $a_1 = \hat{\Phi}'(p=0) = \int_{-\infty}^{\infty} x\Phi(x)dx = 0$ and $a_2 = \frac{1}{2}\hat{\Phi}''(p=0) = \frac{1}{2}\int_{-\infty}^{\infty} x^2\Phi(x)dx = \frac{1}{2}M_2$ where M_2 is the second moment of the dispersal distance PDF $\Phi(x)$ and we have made use of Eq. (4). This approximation corresponds to a diffusive (classical) dispersal. The speed is then from Eq. (5),

$$v = \frac{\alpha}{\tau} \min_{H\tau > a} \sqrt{\frac{\hat{\varphi}(H\tau)}{1 - \hat{\varphi}(H\tau)}} \frac{(H\tau)^{3/2}}{\sqrt{H\tau - a}}, \tag{19}$$

where $\alpha = \sqrt{M_2/2}$ and $a = r\tau$. Let H^* be the value of H where the minimum speed is reached. Then it is found from the second equation in Eq. (5) that

$$\frac{\hat{\varphi}'(H^*\tau)}{\hat{\varphi}(H^*\tau)[1 - \hat{\varphi}(H^*\tau)]} = \frac{3a - 2H^*\tau}{H^*\tau(H^*\tau - a)}, \tag{20}$$

where $\hat{\varphi}'(H^*\tau) = d\hat{\varphi}(H\tau)/d(H\tau)$ evaluated at $H = H^*$. It is easy to see from Eq. (19) that when

$$\hat{\varphi}(H\tau)^{-1} > O(H^2\tau^2) \tag{21}$$

then the speed monotonically decreases and no minimum is reached, which is in accordance with our general condition $M \leqslant N$ found above (here we take $N=2$, so $M \leqslant 2$ is the necessary condition). Let us explain this result for some waiting time distribution of biological interest: Poisson and delta.

The Poisson waiting time distribution is found in the motion of cells when they wait a certain time between successive jumps in order to find the optimum spatial orientation to perform the next jump [10]. The general form for this distribution is

$$\varphi(t) = \frac{1}{\tau\Gamma(m+1)} (t/\tau)^m \exp(-t/\tau), \tag{22}$$

has a maximum at $t = m\tau$, a mean waiting time $\langle t \rangle = \int_0^{\infty} t\varphi(t)dt = \tau(m+1)$, a kurtosis $B_4 = \langle t^4 \rangle / (\langle t^2 \rangle)^2 = (m+3)(m+4)/(m+1)(m+2)$, and a Laplace transform $\hat{\varphi}(H\tau) = (1 + H\tau)^{-m-1}$.

In this case, the speed is given by

$$v_m = \frac{\alpha}{\tau} \min_{H\tau > a} \frac{(H\tau)^{3/2}}{\sqrt{H\tau - a}\sqrt{(1 + H\tau)^{m+1} - 1}} \tag{23}$$

and it is easy to see that for $m > 1$ the speed decreases monotonically to zero and no front minimum speed is reached. Furthermore, the condition (20) has no solution for $m > 1$. To see this, let us express Eq. (20) for this particular case,

$$(m+1)H\tau(H\tau - a)(1 + H\tau)^m = (2H\tau - 3a)[(1 + H\tau)^{m+1} - 1]. \tag{24}$$

We call $R(H\tau)$ and $l(H\tau)$ the right and left hand sides of Eq. (24), respectively. Both sides start from 0 and initially decrease with $H\tau$ but the curve R is below the curve l because $R'(0) = -3a(m+1)$ and $l'(0) = -a(m+1)$. For $H\tau \to \infty$ then $R \sim 2(H\tau)^{m+2}$ and $l \sim (m+1)(H\tau)^{m+2}$. In consequence, both curves intersect if R is above l for large $H\tau$ and this is pos-

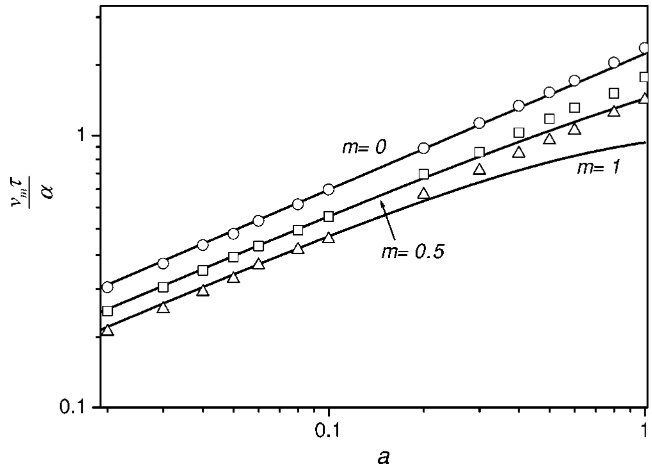

FIG. 1. Plot of the dimensionless front speed $v\tau/\alpha$ versus the dimesionless parameter a for a family of Poisson waiting time PDF's. The speed is calculated from Eq. (23). This plot shows that the speed decreases with m because the waiting time of maximum probability moves to the right when m increases. Values obtained from computer simulations are also shown for $m=0$ (circles), $m=0.5$ (squares), and $m=1$ (triangles).

sible only if $m<1$. This agrees with the condition $M\leqslant 2$ noted above.

In conclusion, when the dispersal is driven by diffusion and a Poisson waiting time PDF, fronts only exist if $m<1$, that is, for a mean waiting time lower than 2τ, or when the time of maximum probability is lower than τ or when the kurtosis of the waiting time PDF is greater than $10/3$.

Two analytical expressions for the speed may be found. For $m=0$ we have the exponentially decaying waiting time distribution $\varphi(t)=\tau^{-1}e^{-t/\tau}$ and the speed of the front is the Fisher one,

$$v_0 = 2\frac{\alpha}{\tau}\sqrt{a}, \tag{25}$$

while for $m=1$ one has $\varphi(t)=\tau^{-1}(t/\tau)\exp(-t/\tau)$, and from Eq. (24) one has the minimum attained at $H^*\tau=4a/(2-a)$ for $a<2$ and the speed is

$$v_1 = 2\frac{\alpha}{\tau}\frac{\sqrt{2}\sqrt{a}}{2+a}, \quad \text{for } a<2. \tag{26}$$

It is easy to see that $v_1<v_0$. For the rest of values $m\in[0,1]$ the minimum must be calculated numerically from Eq. (23). We plot some results for different values of a in Fig. 1; it is seen there that the speed decreases with m because the most probable waiting time moves to the right when m increases. Simulations of these stochastic processes on square lattices have also been performed to prove the validity of the theoretical values of v; they appear in Fig. 1 (points) too. In that simulation, the continuous distributions were discretized in order to adapt them to the lattice (always choosing the parameters of the algorithm in order to minimize the effect of the discretization over the dynamics of the system). The reaction process was introduced by applying the function $f=r\rho F(\rho)$ discussed above to every site at every time step. More details about the simulations can be found in Ref. [6].

The agreement found between these simulations and Eq. (23) is excellent for low values of a. However, when this parameter is higher some discrepancies appear (except for the Fisher's case, $m=0$, which is valid for any a); it probably happens when the divergency in $H^*\tau$ reported above (for example, $a\to 2$ for $m=1$) becomes apparent.

We assume now a waiting time distribution with the form $\varphi(t)=\delta(t-\tau)$ which has a Laplace transform $\hat{\varphi}(\tau H)=e^{-H\tau}$. This PDF models particles which wait a time τ (arbitrarily large) between successive jumps. From Eq. (19) the front speed is given by

$$v = \frac{\alpha}{\tau}\min_{H\tau>a}\frac{(H\tau)^{3/2}}{\sqrt{e^{H\tau}-1}\sqrt{H\tau-a}} \tag{27}$$

and the condition (20) is found to be

$$e^{H\tau} = \frac{3a-2H\tau}{H^2\tau^2-(2+a)H\tau+3a}. \tag{28}$$

The speed must be calculated for $H\tau>a$ but Eq. (28) has no solution because for $H\tau\geqslant a$ the left hand side of Eq. (28) is always greater than the right hand side. So, fronts do not exist for this waiting time distribution. However, for short waiting times fronts may exist. For example, if one takes $e^{H\tau}\simeq 1+H\tau$ one has the parabolic reaction-diffusion equation and Eq. (27) yields again the Fisher speed (25). If one takes now $e^{H\tau}\simeq 1+H\tau+\frac{1}{2}H^2\tau^2$ one has the hyperbolic reaction diffusion [2] where the effect of waiting time τ is stronger than in the parabolic case.

IV. SHORT WAITING TIMES AND LONG DISTANCE OF DISPERSAL

In this section we assume a waiting time PDF with short waiting times and a general jump length PDF. For short waiting times $t\gg\tau$ one may consider the following Taylor expansion: $1/\hat{\varphi}(H)\simeq 1+b_1H$ where $b_1=-\hat{\varphi}'(H=0)/\hat{\varphi}(H=0)^2=\int_0^\infty t\varphi(t)dt\equiv\tau$ is the mean waiting time, so that $1/\hat{\varphi}(H)=1+H\tau$ and Hamilton-Jacobi equation (3) is $H=\tau^{-1}[\hat{\Phi}(p)+a-1]$. Therefore the speed of the front may be calculated from Eq. (5),

$$v = \frac{1}{\tau}\min_{p>0}\frac{\hat{\Phi}(p)+a-1}{p}. \tag{29}$$

It is easy to check that the function $F(p)=[\hat{\Phi}(p)+a-1]/p$ always has a minimum. In the limit $p\to 0$, $F'(p)=\{\hat{\Phi}'(p)/p-[\hat{\Phi}(p)+a-1]/p^2\}\sim -1/p^2<0$ and the speed is a decreasing function of p when p is near 0. On the other hand, in the limit $p\to\infty$, $F'(p)\sim\hat{\Phi}'(p)/p>0$ and the speed is an increasing function of p. In consequence, the speed always reaches a positive minimum value in the interval $p\in[0,\infty)$ and the front always exists. This is in accordance

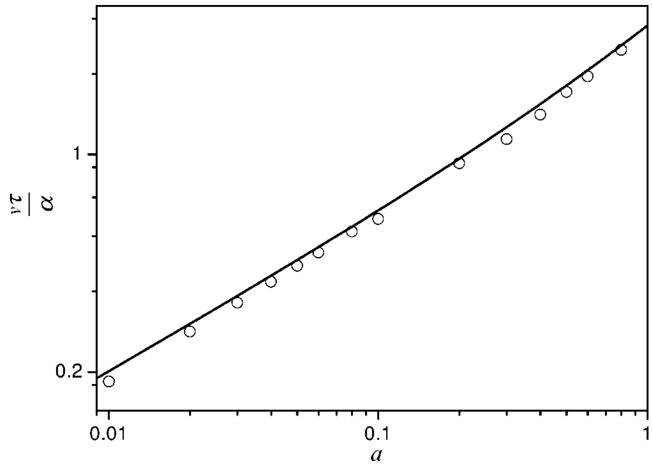

FIG. 2. Plot of the dimensionless front speed $v\tau/\alpha$ versus the dimensionless parameter a for a Laplace dispersal distance PDF. The theoretical speed (line) is calculated from Eq. (33) and compared to simulations (points).

with our general arguments above: here $M=1$, so any value for N is allowed.

The speed is finally computed from Eq. (6)

$$v = \frac{1}{\tau}\frac{\hat{\Phi}(p^*) + a - 1}{p^*}, \tag{30}$$

where p^* is solution of $\hat{\Phi}(p^*)+a-1=p^*\hat{\Phi}'(p^*)$.

We report here the Laplace kernel

$$\Phi(x) = \frac{1}{2\alpha}e^{-x/\alpha} \tag{31}$$

as a typical example of a kernel usual in ecological systems related to dispersal of species [5]. When the Laplace transform of Eq. (31),

$$\hat{\Phi}(p) = (1-\alpha^2p^2)^{-1}, \tag{32}$$

is introduced into Eq. (29), the resulting expression we obtain for the velocity is

$$v = \frac{2\alpha}{\tau}\frac{4(1-a)^2}{(3-\sqrt{1+8a})^2}\left[\frac{-1-2a+\sqrt{1+8a}}{2(1-a)}\right]^{1/2}. \tag{33}$$

In Fig. 2 we show the comparison between this theoretical prediction for v and the values found from direct simulations of the stochastic process as described above.

V. CONCLUSIONS

Traveling wave front solutions of reaction-dispersal processes with a minimum speed selected do not always exist for an arbitrary waiting time and dispersal distance distributions. The existence of these fronts depends on the PDF's chosen to model the process, so the main conclusion of this work is that in general they cannot be chosen separately in order to have traveling wave fronts selecting the minimum speed. Specifically, we have achieved that traveling wave solutions with a minimum speed selected are only possible if the order in the truncation of the Taylor expansion of the inverse of the Laplace transform of the waiting time PDF is not higher than the order in the truncation of the Taylor expansion of the bilateral transform of the dispersal distance PDF. Then, if we are restricted to observe what happens at long times we cannot obtain information from what happens at short distances and so no minimum speed is selected. However, when our measurements are restricted to long dispersal distances then we find that no restrictions, on the truncation of the Taylor expansion of the inverse of the Laplace transform of the waiting time PDF, appear for the existence of fronts, as both short and long waiting times are tractable.

As some illustrative examples, we have seen that when the dispersal process is diffusive, the existence condition requires that the inverse of the Laplace transform of the waiting time PDF may be expanded in Taylor up to second order (not more). This result has been verified for different waiting time PDF's and contrasted with numerical simulations. For a family of Poisson PDF's we have shown that traveling wave front solutions with a minimum speed selected exist only if $m\leqslant 1$. Likewise, for a single waiting time PDF these fronts only exist if the waiting time is sufficiently small. However, when the dispersal process is described by a long-tailed dispersal distance PDF traveling wave front solution with a minimum speed selected exist if the waiting time is sufficiently small. We have detailed this case for the Laplace dispersal distance PDF and it has been compared to numerical simulations.

So, we think that the work presented here can be useful for a wide number of scientists working on reaction-dispersal systems where the specific pattern of dispersal needs to be described in great detail. We have reported some specific examples in relation to ecological systems, but many other potential applications, covering very different areas [1,11], can benefit from these results.

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