Scaling and renormalization for the Kolmogorov-Petrovskii-Piskunov equation with turbulent convection

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The problem of determining the upper bounds for the ensemble-averaged reaction front position and speed in a fully developed *three-dimensional* turbulent flow has been examined, in which the reaction is of Kolmogorov-Petrovskii-Piskunov type and turbulent velocity is a Gaussian random field exhibiting long-range correlations and infrared divergence in the limit of large Reynolds number. An asymptotic method has been developed that gives the general formalism for determining the upper bounds for reaction front in the long-time, large-distance limit. Two anomalous scaling regimes and corresponding scaling functions have been determined by the use of exact renormalization procedure. [S1063-651X(97)03503-4]

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I. INTRODUCTION

In recent years there has been a substantial theoretical interest in the study of reaction front propagation in turbulent flows (see, for example, [1–3], and references therein). The main quantities of interest are the large scale turbulent flame speed and its parametric dependence on the statistical characteristics of turbulent flow. The importance of analytic procedures for determining turbulent flame velocity has long been recognized because the direct numerical calculation of this quantity is usually very expensive and time consuming, especially if we are interested in the enhanced flame speed in a high Reynolds number turbulent flow [4–7].

An important starting point to deal with the problem of turbulent reaction front propagation is the Kolmogorov-Petrovskii-Piskunov (KPP) equation with random convection term. Recently a great deal of progress has been made in this theory using partial differential equation (PDE) techniques for viscosity solutions for Hamilton-Jacobi equations [8–10] and a rigorous renormalization procedure [11–13], but here the results are restricted to a rather simple case of two separated length scales for the velocity field or two-dimensional shear flow with many spatiotemporal scales.

In this paper we attempt to remove these restrictions, by considering *three-dimensional* random velocity field with arbitrarily many spatial and temporal scales. The main purpose is to derive an equation determining an upper bound for ensemble-averaged reaction front position and speed and thereby to provide the framework for a detailed study of how the random velocity field with long-range correlations in space and time may influence front propagation in the long-time, large-distance limit. Our method of analysis introduces a representation of the solution of the KPP equation with random convection in terms of functional integrals with respect to Markov processes [11–15]. We employ a singular perturbation method involving small values of the ratio of the Kolmogorov length scale to the integral length scale of turbulent flow. We also use the exact renormalization theory for turbulent transport (while the main stages of this theory will be given in what follows we refer to Avellaneda and Majda for a complete and careful discussion and further references [16,17]).

The main area of practical interest concerning reaction front propagation in a random velocity field is a turbulent combustion in the so-called flamelet regime [4–7]. Although the theory based on the KPP equation may not be directly applicable to premixed flames (see, however, [18,19]), it does provide a reference point for interpreting the experimental results on the acceleration of reaction front [7,12].

II. KOLMOGOROV-PETROVSKII-PISKUNOV EQUATION WITH RANDOM CONVECTION

Consider a nondimensional scalar field $\varphi(t,\mathbf{x})$ whose dynamical evolution is governed by the Kolmogorov-Petrovskii-Piskunov equation with random convection term

$$\frac{\partial\varphi}{\partial t}+\mathbf{v}(t,\mathbf{x})\cdot\nabla\varphi=D\nabla^2\varphi+c(\epsilon|\mathbf{x}|,\varphi)\varphi,\quad \mathbf{x}\in\mathbb{R}^3 \qquad (1)$$

where the velocity $\mathbf{v}(t,\mathbf{x})$ is assumed to be an incompressible homogeneous isotropic Gaussian random field with zero mean and the reaction rate $c(\epsilon|\mathbf{x}|,\varphi)\varphi$ is of Kolmogorov-Petrovskii-Piskunov type,

$$c(\epsilon|\mathbf{x}|,0)=\max_{\varphi\in[0,1]}c(\epsilon|\mathbf{x}|,\varphi)>0\,,\quad c(\epsilon|\mathbf{x}|,1)=0\,. \qquad (2)$$

Here the space and time variables are rescaled so that space is measured in units of the Kolmogorov length scale $\eta=(\nu^3/\overline{\varepsilon})^{1/4}$, and time is measured in units of $t_k=\eta/v_k$. The Kolmogorov velocity scale $v_k=(\nu\overline{\varepsilon})^{1/4}$ is used as unit of random velocity field; $\overline{\varepsilon}$ is the average dissipation rate of energy per unit mass. We assume that the function $c(\epsilon|\mathbf{x}|,\varphi)$ varies on the integral scale of turbulence and depends only on the distance from the point $\mathbf{x}=\mathbf{0}$; that is why the reaction rate c involves a small parameter $\epsilon=\mathrm{Re}^{-3/4}$, the ratio of the Kolmogorov length scale η to the integral length scale l_0, $\mathrm{Re}=u_0l_0/\nu$ is a Reynolds number. If the diffusion coefficient D nondimensionalized by the viscosity

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ν and the parameter $c=c(\epsilon|\mathbf{x}|,0)=\text{const}$ are of order of unity, then the ''laminar'' flame velocity $v_L=(2cD)^{1/2}$ is of the order of the Kolmogorov velocity v_k and the ''laminar'' flame thickness $l_L=(D/c)^{1/2}$ is of the order of the Kolmogorov dissipation length η.

The initial condition is assumed to be spherically symmetrical, namely,

$$\varphi(0,\mathbf{x})=\varphi_0(\epsilon|\mathbf{x}|)=\begin{cases}1 & \text{if } \epsilon|\mathbf{x}|<r\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

These data mean that at some initial time, a spherical ''hot'' pocket with $\varphi=1$ is introduced into an infinite ''cold'' medium with $\varphi=0$. It is assumed that the initial dimension radius of the ''hot'' pocket is rl_0, where l_0 is the integral scale of turbulence.

Since the velocity $\mathbf{v}(t,\mathbf{x})$ is a Gaussian field with zero mean, its statistical properties are determined by the *model* correlation tensor [20]

$$\langle v_i(t,\mathbf{x})v_j(\tau,\mathbf{y})\rangle=B_{ij}(|t-\tau|,\mathbf{x}-\mathbf{y}), \tag{4}$$

where

$$B_{ij}(|t-\tau|,\mathbf{x}-\mathbf{y})=\frac{1}{4\pi}\int\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\omega(t-\tau)}E(k,\omega)k^{-2}\times\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)d\mathbf{k}d\omega \tag{5}$$

and

$$E(k,\omega)=\frac{2(\epsilon^2+k^2)^{-1/3+z/2}}{\pi[1+\omega^2(\epsilon^2+k^2)^{-2/3+z}]}E(k),$$

$$E(k)=\frac{V^2k^2\exp(-k^2)}{(\epsilon^2+k^2)^{11/6-\sigma/2}}. \tag{6}$$

This spectral representation involves three important parameters ϵ, σ, and z [12,16,17]. The small parameter ϵ is the ratio of the Kolmogorov length scale to the integral length scale [see also Eq. (3)]. The spectral exponent σ appearing here may be thought of as representing a deviation of the energy spectrum from the Kolmogorov-Obukhov one in the inertial range, where as follows from Eq. (6) $E(k)\sim k^{-5/3+\sigma}$ as $\epsilon\ll k\ll 1$. The specific feature of the energy spectrum $E(k)$ in Eq. (6) is the infrared divergence of the kinetic energy in the limit of high Reynolds number (Re$\to\infty$ or $\epsilon\to 0$)

$$\frac{1}{2}\langle\mathbf{v}^2\rangle=\int_0^\infty E(k)dk\sim\epsilon^{-2/3+\sigma}\quad\text{as}\quad\epsilon\to 0\,. \tag{7}$$

It follows from Eq. (7) that the parameter σ can also be interpreted as a measure of the infrared divergence. The dynamic exponent z describes the dependence of the correlation time $(\varepsilon^2+k^2)^{-1/3+z/2}$ upon k. For $z=2/3$, all ''eddies'' have identical turnover time. It is clear that the Kolmogorov-Obukhov turbulence corresponds to the case in which $\sigma=0$ and $z=0$ [20].

III. ENSEMBLE-AVERAGED UPPER BOUNDS ON REACTION FRONT POSITION AND SPEED

In this paper we are concerned with a reaction front propagation from the ''hot'' pocket (3) in the limit Re$\to\infty$, that is, $\epsilon\to 0$. A specific intention is to determine the upper bounds for the ensemble-averaged reaction front position and speed in the long-time, large-distance limit.

Before we consider this problem, it is instructive to show how a reaction front arises naturally as a result of the scaling procedure for a special case of initial value problem (1)–(3) when the convective term is absent, i.e., $\mathbf{v}(t,\mathbf{x})=\mathbf{0}$. Here we follow the method developed by Freidlin [14,15].

The basic idea is an introduction of the small parameter ϵ and rescaling the space and time in such a way that the rescaled solution of Eqs. (1)–(3) with $\mathbf{v}=\mathbf{0}$, that is, $\varphi^\epsilon(t,\mathbf{x})=\varphi(t/\epsilon,\mathbf{x}/\epsilon)$, takes only two values 0 and 1 as $\epsilon\to 0$. This means that the entire problem of finding the wave solution of Eqs. (1)–(3) can be reduced to the analysis of the evolution of reaction front separating the area where $\varphi^\epsilon(t,\mathbf{x})\to 0$ and the area where $\varphi^\epsilon(t,\mathbf{x})\to 1$ [14,15].

The initial value problem for the rescaled field $\varphi^\epsilon(t,\mathbf{x})$ takes the form

$$\frac{\partial\varphi^\epsilon}{\partial t}=\epsilon D\nabla^2\varphi^\epsilon+\frac{1}{\epsilon}c(\varphi^\epsilon)\varphi^\epsilon,\quad\varphi^\epsilon(0,\mathbf{x})=\varphi_0(|\mathbf{x}|). \tag{8}$$

The solution of this problem can be found from the functional equation [14,15]

$$\varphi^\epsilon(t,\mathbf{x})=\boldsymbol{\mathcal{E}}\varphi_0(|\mathbf{x}+(2\epsilon D)^{1/2}\boldsymbol{\mathcal{W}}(t)|)\times\exp\left\{\frac{1}{\epsilon}\int_0^t c(\varphi^\epsilon(t-s,\mathbf{x}+(2\epsilon D)^{1/2}\boldsymbol{\mathcal{W}}(s)))ds\right\}, \tag{9}$$

where $\boldsymbol{\mathcal{E}}$ denotes the expectation over the three-dimensional Wiener process $\boldsymbol{\mathcal{W}}(s)$.

It follows from the inequality $c(\varphi^\epsilon)\leqslant c(0)$, Eq. (3), and Eq. (9) that

$$\varphi^\epsilon(t,\mathbf{x})\leqslant\frac{1}{(4\pi\epsilon Dt)^{3/2}}\exp\left(\frac{c(0)t}{\epsilon}\right)\times\int_{|\mathbf{x}+\mathbf{z}|<r}\exp\left(-\frac{\mathbf{z}^2}{4\epsilon Dt}\right)d\mathbf{z}\sim\exp\left(\frac{G(t,\mathbf{x})}{\epsilon}\right), \tag{10}$$

where

$$G(t,\mathbf{x})=c(0)t-\frac{(x-r)^2}{4Dt}\Theta(x-r)$$

and Θ is a Heaviside function.

It is evident that $\varphi^\epsilon(t,\mathbf{x})\to 0$ if $G(t,\mathbf{x})<0$ and $\epsilon\to 0$. One can also find that $\varphi^\epsilon(t,\mathbf{x})\to 1$ if $G(t,\mathbf{x})>0$ and $\epsilon\to 0$. One may conclude that the equation $G(t,\mathbf{x})=0$ determines the position of reaction front $x(t)=r+\sqrt{4c(0)D}t$ and speed $dx/dt=\sqrt{4c(0)D}$.

Now let us turn to the full problem (1)–(3). The main task is to extend the simple analysis given above to the random-advection problem with a view to determining the ensemble-averaged characteristics of the large scale reaction front for

Eqs. (1)–(3), in particular, the upper bounds for reaction front position and speed. It follows from the results developed in [11–13] that such bounds can be described through the effective equations with the parameters depending on the statistical characteristics of the random velocity field $\mathbf{v}(t,\mathbf{x})$.

In order to find the upper bound position for ensemble-averaged reaction front

$$S_G=\{\mathbf{x}\in\mathbb{R}^3:\quad G(t,\mathbf{x})=0\},$$

we need to determine the effective function $G(t,\mathbf{x})$ [11,12],

$$G(t,\mathbf{x})=\lim_{\epsilon\to 0}\lambda(\epsilon)\ln\left\langle \varphi^*\left(\frac{t}{\lambda(\epsilon)},\frac{\mathbf{x}}{\epsilon}\right)\right\rangle. \tag{11}$$

Here and throughout this paper the angular brackets $\langle\ \rangle$ denote the ensemble averaging over velocity statistics and φ^* is a solution of Eqs. (1)–(3) when $c(\varphi)$ is replaced by its maximum value $c(0)$. To get uniformly valid result independent of Reynolds number in the limit $\epsilon\to 0$ ($\mathrm{Re}\to\infty$) we have to find a rescaling function $\lambda(\epsilon)$ with $\lim_{\epsilon\to 0}\lambda(\epsilon)=0$ that makes the limit (11) nontrivial.

It follows from Eq. (11) and the inequality $\varphi<\varphi^*$ that

$$\lim_{\epsilon\to 0}\left\langle \varphi\left(\frac{t}{\lambda},\frac{\mathbf{x}}{\epsilon}\right)\right\rangle = 0 \quad \text{if} \quad G(t,\mathbf{x})<0 \tag{12}$$

and therefore $G(t,\mathbf{x})=0$ may be regarded as an equation determining the upper bound position for ensemble-averaged reaction front.

We first find an explicit expression for the rescaled function $\varphi(t/\lambda,\mathbf{x}/\epsilon)$ by using a functional integral technique.

Let us introduce new variables, denoted by primes, by

$$t'=\lambda(\epsilon)t,\quad \mathbf{x}'=\epsilon\mathbf{x}. \tag{13}$$

Substitution of Eq. (13) into Eq. (1) gives the equation for

$$\varphi^\epsilon(t',\mathbf{x}')=\varphi\left(\frac{t'}{\lambda},\frac{\mathbf{x}'}{\epsilon}\right), \tag{14}$$

namely,

$$\frac{\partial\varphi^\epsilon}{\partial t}+\frac{\epsilon}{\lambda}\mathbf{v}\left(\frac{t}{\lambda},\frac{\mathbf{x}}{\epsilon}\right)\nabla\varphi^\epsilon=\frac{\epsilon^2 D}{\lambda}\nabla^2\varphi^\epsilon+\frac{1}{\lambda}c(|\mathbf{x}|,\varphi^\epsilon)\varphi^\epsilon, \tag{15}$$

with the initial condition

$$\varphi^\epsilon(0,\mathbf{x})=\varphi_0(|\mathbf{x}|), \tag{16}$$

where for convenience we have omitted the primes.

The solution $\varphi^\epsilon(t,\mathbf{x})$ of the Cauchy problem (15), (16) obeys the functional integral equation [14,15]

$$\varphi^\epsilon(t,\mathbf{x})=\mathcal{E}\varphi_0(|\mathbf{x}+\mathbf{z}(t)|)\exp\left\{\frac{1}{\lambda}\int_0^t c(|\mathbf{x}+\mathbf{z}(s)|,\varphi^\epsilon(t-s,\mathbf{x}+\mathbf{z}(s)))ds\right\}, \tag{17}$$

where $\mathcal{E}$ denotes the expectation over the trajectories $\mathbf{z}(s)$ that are the solution of the stochastic differential equations

$$d\mathbf{z}(s)=-\frac{\epsilon}{\lambda}\mathbf{v}\left(\frac{t-s}{\lambda},\frac{\mathbf{x}+\mathbf{z}(s)}{\epsilon}\right)ds+\left(\frac{2\epsilon^2 D}{\lambda}\right)^{1/2}d\boldsymbol{\mathcal{W}}(s),\quad \mathbf{z}(0)=\mathbf{0},\ 0\le s\le t. \tag{18}$$

Here $\boldsymbol{\mathcal{W}}(s)$ is the standard three-dimensional Wiener process with a probability density functional of the form

$$P[\boldsymbol{\mathcal{W}}]=\exp\left\{-\frac{1}{2}\int\left(\frac{d\boldsymbol{\mathcal{W}}}{ds}\right)^2 ds\right\}. \tag{19}$$

In order to progress any further with the formula (17) it is necessary to have an explicit form for the mean value. It may be written in the form

$$\varphi^\epsilon(t,\mathbf{x})=\int\varphi_0(|\mathbf{x}+\mathbf{z}(t)|)\exp\left\{\frac{1}{\lambda}\int_0^t c(|\mathbf{x}+\mathbf{z}(s)|,\varphi^\epsilon(t-s,\mathbf{x}+\mathbf{z}(s)))ds\right\}P[\mathbf{z}(s)]\mathcal{D}\mathbf{z}(s), \tag{20}$$

where the probability density functional $P[\mathbf{z}(s)]$ for the random process $\mathbf{z}(s)$ may be readily found from Eqs. (18) and (19),

$$P[\mathbf{z}(s)]=J\ \exp\left\{-\frac{\lambda}{4\epsilon^2 D}\int_0^t\left[\frac{d\mathbf{z}}{ds}+\frac{\epsilon}{\lambda}\mathbf{v}\left(\frac{t-s}{\lambda},\frac{\mathbf{x}+\mathbf{z}(s)}{\epsilon}\right)\right]^2 ds\right\}. \tag{21}$$

The functional integration in Eq. (20) is performed over all trajectories $\mathbf{z}(s)$ starting at $s=0$ with $\mathbf{z}(0)=0$. It should be noted that the integral equation (20) is valid for any fixed realization of the random velocity field $\mathbf{v}$. The Jacobian $J=\exp(\alpha\int_0^t\boldsymbol{\nabla}\cdot\mathbf{v}ds)$ corresponding to mapping $\mathbf{z}(s)$ onto $\boldsymbol{\mathcal{W}}(s)$ is equal to unity for the incompressible fluid ($\boldsymbol{\nabla}\cdot\mathbf{v}=0$).

Since we are concerned with an average value of $\varphi^\epsilon(t,\mathbf{x})$ over velocity statistics, it is convenient to make the functional integral (20), (21) ''linear'' in the velocity field. By using an auxiliary vector $\mathbf{p}(s)$, we replace the Lagrangian functional integral (20), (21) in which the random velocity $\mathbf{v}$ appears in quadratic form by its Hamiltonian version [21,22]. It follows from Eqs. (20), (21) and the formula

$$\int \exp\left\{ i\int a(s)p(s)ds \right\} \exp\left\{ -b\int p^2(s)ds \right\} \mathcal{D}p(s) = \exp\left\{ -\frac{1}{4b}\int a^2(s)ds \right\}$$

that

$$\varphi^{\epsilon}(t,\mathbf{x}) = \int\int \varphi_0(|\mathbf{x}+\mathbf{z}(t)|) \exp\left\{ \frac{1}{\lambda}\int_0^t \left[c(|\mathbf{x}+\mathbf{z}(s)|, \varphi^{\epsilon}(t-s,\mathbf{x}+\mathbf{z}(s))) + \lambda i\mathbf{p}(s)\cdot\frac{d\mathbf{z}}{ds} - \epsilon^2 D\mathbf{p}^2(s) \right.\right.$$
$$\left.\left. + \epsilon i\mathbf{p}(s)\cdot\mathbf{v}\left(\frac{t-s}{\lambda}, \frac{\mathbf{x}+\mathbf{z}(s)}{\epsilon}\right)\right] ds \right\} \mathcal{D}\mathbf{z}(s)\mathcal{D}\mathbf{p}(s). \tag{22}$$

We are now in a position to find an ensemble average of $\varphi^*(t,\mathbf{x})$ appearing in the formula (11). Using the well known formula for the Gaussian variable ξ with zero mean,

$$\langle \exp(a\xi)\rangle = \exp\left\{ \frac{a^2\langle \xi^2\rangle}{2} \right\},$$

we obtain

$$\left\langle \exp\left\{ \frac{i\epsilon}{\lambda}\int_0^t \mathbf{p}(s)\cdot\mathbf{v}\left(\frac{t-s}{\lambda}, \frac{\mathbf{x}+\mathbf{z}(s)}{\epsilon}\right) ds \right\} \right\rangle = \exp\left\{ -\frac{\epsilon^2}{2\lambda^2}\int_0^t\int_0^t \sum_{i,j=1}^{3} p_i(s_1)p_j(s_2) \right.$$
$$\left. \times \left\langle v_i\left(\frac{t-s_1}{\lambda}, \frac{\mathbf{x}+\mathbf{z}(s_1)}{\epsilon}\right) v_j\left(\frac{t-s_2}{\lambda}, \frac{\mathbf{x}+\mathbf{z}(s_2)}{\epsilon}\right)\right\rangle ds_1 ds_2 \right\}. \tag{23}$$

It follows from this result and Eq. (22) that the average value of $\varphi^*(t,\mathbf{x})$ may be written as

$$\langle \varphi^*(t,\mathbf{x})\rangle = \int\int \varphi_0(|\mathbf{x}+\mathbf{z}(t)|) \exp\left\{ \frac{1}{\lambda}\int_0^t \left(c(|\mathbf{x}+\mathbf{z}(s)|) + \lambda i\mathbf{p}(s)\cdot\frac{d\mathbf{z}}{ds} - \epsilon^2 D\mathbf{p}^2(s) \right) ds \right.$$
$$\left. - \frac{\epsilon^2}{2\lambda^2}\int_0^t\int_0^t \sum_{i,j=1}^{3} p_i(s_1)p_j(s_2)B_{ij}\left(\frac{|s_1-s_2|}{\lambda}, \frac{\mathbf{z}(s_1)-\mathbf{z}(s_2)}{\epsilon}\right) ds_1 ds_2 \right\} \mathcal{D}\mathbf{z}(s)\mathcal{D}\mathbf{p}(s). \tag{24}$$

Replacing $\mathbf{p}(s)$ by $i\mathbf{u}(s)/\lambda$ and assuming that the vector $\mathbf{u}(s)$ is real, we can rewrite Eq. (24) in the Hamiltonian form (one can find a detailed discussion concerning this replacement in [23])

$$\langle \varphi^*(t,\mathbf{x})\rangle = \int\int \varphi_0(|\mathbf{x}+\mathbf{z}(t)|) \exp\left\{ -\frac{1}{\lambda}\int_0^t \mathbf{u}\cdot\frac{d\mathbf{z}}{ds}ds + \frac{1}{\lambda}H^{\epsilon} \right\} \mathcal{D}\mathbf{z}(s)\mathcal{D}\left(\frac{i\mathbf{u}(s)}{\lambda}\right), \tag{25}$$

where the Hamiltonian functional H^{ϵ} has a form

$$H^{\epsilon}[\mathbf{z}(s),\mathbf{u}(s)] = \int_0^t \left(c(|\mathbf{x}+\mathbf{z}(s)|) + \frac{\epsilon^2 D}{\lambda^2}\mathbf{u}^2 \right) ds + \frac{\epsilon^2}{2\lambda^3}\int_0^t\int_0^t \sum_{i,j=1}^{3} u_i(s_1)u_j(s_2)B_{ij}\left(\frac{|s_1-s_2|}{\lambda}, \frac{\mathbf{z}(s_1)-\mathbf{z}(s_2)}{\epsilon}\right) ds_1 ds_2. \tag{26}$$

These formulas together with Eq. (11) allow us to find the effective function $G(t,\mathbf{x})$ and thereby the upper bound for the ensemble-averaged reaction front position $S_G = \{\mathbf{x}\in\mathbb{R}^3 : G(t,\mathbf{x}) = 0\}$ without directly solving the nonlinear problem (1)–(3).

The asymptotic behavior of $\langle \varphi^*(t,\mathbf{x})\rangle$ in the limit $\epsilon\to 0$ can be obtained by the saddle-point approximation

$$\langle \varphi^*(t,\mathbf{x})\rangle \sim \exp\frac{1}{\lambda}\max\left\{ -\int_0^t \mathbf{u}\cdot\frac{d\mathbf{z}}{ds}ds + H^0 :\ \mathbf{z}(0)=\mathbf{0}, |\mathbf{x}+\mathbf{z}(t)| = r,\ |\mathbf{x}+\mathbf{z}(s)| > r \right\}.$$

Substitution of this in Eq. (11) yields

$$G(t,\mathbf{x}) = \max\left\{ -\int_0^t \mathbf{u}\cdot\frac{d\mathbf{z}}{ds}ds + H^0 :\ \mathbf{z}(0)=\mathbf{0},\ |\mathbf{x}+\mathbf{z}(t)| = r,\ |\mathbf{x}+\mathbf{z}(s)| > r \right\} \tag{27}$$

provided that $H^0 = \lim_{\epsilon\to 0} H^{\epsilon}$ has a nontrivial limit.

It is clear that functions $\mathbf{z}_*(s)$ and $\mathbf{u}_*(s)$ that maximize the functional in Eq. (27) can be found from the Hamilton variational equations

$$\frac{d\mathbf{z}_*}{ds}=\frac{\delta H^0}{\delta \mathbf{u}_*},\quad \frac{d\mathbf{u}_*}{ds}=-\frac{\delta H^0}{\delta \mathbf{z}_*}.$$

Thus the problem of finding the upper bound position for large scale front has been reduced to maximizing the functional $-\int_0^t \mathbf{u}\cdot(d\mathbf{z}/ds)ds+ H^0$ and finding the effective Hamiltonian $H^0=\lim_{\epsilon\to 0}H^\epsilon$.

IV. RENORMALIZATION PROCEDURE

We can expect from the results obtained in [12,16,17] that the limit $\epsilon\to 0$ might exhibit a wide range of different scaling phenomena in the behavior of the Hamiltonian functional H^ϵ when the spectral parameters σ and z are varied. Here we develop the explicit expressions for H^0 by using the exact renormalization theory [16,17].

First, it is convenient to perform the rescaling of the wave vector $\mathbf{k}$ and frequency ω as follows:

$$\omega\to\lambda(\epsilon)\omega,\quad \mathbf{k}\to\epsilon\mathbf{k}.$$

Under this transformation the Hamiltonian functional H^ϵ changes to

$$H^\epsilon=\int_0^t\left(c(|\mathbf{x}+\mathbf{z}(s)|)+\frac{\epsilon^2 D}{\lambda^2}\mathbf{u}^2\right)ds+\frac{V^2\epsilon^{2/3+\sigma+z}}{4\pi^2\lambda^2}\int_0^t\int_0^t\sum_{i,j=1}^{3}u_i(s_1)u_j(s_2)\int_{\mathbf{k}}\int_{\omega}e^{i\mathbf{k}\cdot[\mathbf{z}(s_1)-\mathbf{z}(s_2)]-i\omega(s_1-s_2)}$$
$$\times\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{(1+k^2)^{-13/6+z/2+\sigma/2}\exp(-\epsilon^2k^2)}{1+\lambda^2\epsilon^{-4/3+2z}\omega^2(1+k^2)^{-2/3+z}}d\mathbf{k}d\omega ds_1 ds_2. \tag{28}$$

Now we are in a position to find the limit $H^0=\lim_{\epsilon\to 0}H^\epsilon$ that must be uniformly valid independent of the Reynolds number Re. The last requirement gives us the time rescaling function $\lambda(\epsilon)$. In what follows we consider only two distinct scaling regimes for which the phenomenon of infrared divergence plays the main role. It is easy to show that in these cases the simple diffusive scaling $\lambda(\epsilon)=\epsilon$ leads to the divergence of H^ϵ as $\epsilon\to 0$. The exact renormalization procedure amounts to a choice of the appropriate scaling functions $\lambda(\epsilon)$ to avoid this singular behavior. The more complicated cases involving random nonlocal diffusivity [16,17] will not be treated here.

(1) *Rapid correlation time.* Consider the case when

$$\lambda^2\epsilon^{-4/3+2z}\to 0\quad \text{as}\quad \epsilon\to 0\,. \tag{29}$$

Here we show that in the limit (29) the upper bounds can be obtained *exactly*. The first step is to choose $\lambda(\epsilon)$ such that the limit (9) is bounded and nontrivial. The above results show that a sufficient condition for the Hamiltonian (28) to remain bounded as $\epsilon\to 0$ is that $\epsilon^{2/3+\sigma+z}/\lambda^2=1$, and hence

$$\lambda(\epsilon)=\epsilon^{1/3+\sigma/2+z/2} \tag{30}$$

provided $\sigma+3z>2/3$. If we consider the region where $\sigma+3z>2/3$ and $4/3>\sigma+z$, then molecular diffusion is negligible because $\epsilon/\lambda\to 0$ as $\epsilon\to 0$.

It is evident from Eq. (28) that in the limit (29) the spectral density loses its dependence on the frequency ω as ϵ tends to zero and this implies that the scaling procedure together with limit $\epsilon\to 0$ generates a white noise in time. Therefore the next step is to use the identity

$$\frac{1}{2\pi}\int\exp[i\omega(s_1-s_2)]d\omega=\delta(s_1-s_2)$$

that allows us to find a relatively simple expression for H^0,

$$H^0=\int_0^t[c(|\mathbf{x}+\mathbf{z}(s)|)+D_T\mathbf{u}^2(s)]ds,$$

$$D_T=\frac{4}{3}V^2\int_0^\infty k^2(1+k^2)^{-13/6+\sigma/2+z/2}dk. \tag{31}$$

In particular, when $c=\text{const}$, we have

$$G(t,\mathbf{x})=ct-\min\left\{\int_0^t\left(\mathbf{u}\cdot\frac{d\mathbf{z}}{ds}-D_T\mathbf{u}^2\right)ds:\ \mathbf{z}(0)=\mathbf{0},\ |\mathbf{x}+\mathbf{z}(t)|=r,\ |\mathbf{x}+\mathbf{z}(s)|>r\right\}. \tag{32}$$

It is not difficult to find from Eq. (32) that

$$G(t,\mathbf{x})=ct-\frac{(x-r)^2}{4D_Tt}.$$

It is clear that the upper bound $S_G=\{\mathbf{x}\in\mathbb{R}^3:\ G(t,\mathbf{x})=0\}$ corresponds to the outgoing spherical front with the position $x(t)=r+\sqrt{4c(0)D_T}t$ and constant velocity $\sqrt{4cD_T}$.

The results of our coarse-graining procedure allow us to conclude that in the limit (29) the large scale behaviors of the full problem (1)–(3) and its simplified version (8) are the same. The physical reason for this is that under the condition (29) the renormalization procedure generates a white noise in time and therefore the large scale and long-time turbulent transport can be described by a conventional diffusion model with the effective eddy diffusivity D_T. It should be noted that the case of ''rapid correlation time'' is the analog of ''region II'' of Avellaneda and Majda [16,17,24] in a sense that here the time-decorrelation effects play the crucial role.

(2) *Frozen turbulence limit.* This limit corresponds to the case when

$$\lambda^2\epsilon^{-4/3+2z}\to\infty \quad \text{as} \quad \epsilon\to 0. \tag{33}$$

Unlike the case of ''rapid correlation time'' considered above, no simple expression for the effective Hamiltonian H^0 can be obtained here. It should also be noted that this limit is the analog of ''region III'' in [24].

It is convenient first to perform the integration with respect to the frequency ω. Simple calculations give

$$H^\epsilon=\int_0^t\left(c(|\mathbf{x}+\mathbf{z}(s)|)+\frac{\epsilon^2 D}{\lambda^2}\mathbf{u}^2(s)\right)ds+\frac{V^2\epsilon^{4/3+\sigma}}{8\pi\lambda^3}\int_0^t\int_0^t\sum_{i,j=1}^{3}u_i(s_1)u_j(s_2)$$
$$\times\int_{\mathbf{k}}e^{i\mathbf{k}\cdot[\mathbf{z}(s_1)-\mathbf{z}(s_2)]}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)e^{-\lambda^{-1}\epsilon^{2/3-z}(1+k^2)^{1/3-z/2}|s_1-s_2|}(1+k^2)^{-11/6+\sigma/2}e^{-\epsilon^2k^2}d\mathbf{k}ds_1ds_2. \tag{34}$$

Here the factor $\lambda^{-1}\epsilon^{2/3-z}(1+k^2)^{1/3-z/2}$ in front of $|s_1-s_2|$ defines the inverse k-dependent correlation time that continues to decrease as $\epsilon\to 0$ [see Eq. (33)]. This is a reason the limit (33) may be regarded as a ''frozen turbulence limit.''

In order for the functional H^ϵ to converge as $\epsilon\to 0$ we must have $\epsilon^{4/3+\sigma}/\lambda^3=1$, that is,

$$\lambda(\epsilon)=\epsilon^{4/9+\sigma/3} \tag{35}$$

provided $\sigma+3z<2/3$. The range of σ for which the molecular diffusion becomes irrelevant is $\sigma<5/3$.

With this scaling, we compute

$$H^0=\int_0^t c(|\mathbf{x}+\mathbf{z}(s)|)ds+\frac{V^2}{8\pi}\int_0^t\int_0^t\int_{\mathbf{k}}\sum_{i,j=1}^{3}u_i(s_1)u_j(s_2)e^{i\mathbf{k}\cdot[\mathbf{z}_1(s_1)-\mathbf{z}_2(s_2)]}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(1+k^2)^{-11/6+\sigma/2}d\mathbf{k}ds_1ds_2. \tag{36}$$

The last result may be interpreted as follows. Asymptotically we have the front propagation in the effective stationary Gaussian random field $\mathbf{v}_{\text{eff}}(\mathbf{x})$ with correlation tensor

$$B_{ij}(\mathbf{x}-\mathbf{y})=\frac{V^2}{4\pi}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(1+k^2)^{-11/6+\sigma'2}d\mathbf{k}. \tag{37}$$

The function $G(t,\mathbf{x})$ determining the upper bound position can be written as

$$G(t,\mathbf{x})=\max\left\{\int_0^t\left(c(|\mathbf{x}+\mathbf{z}(s)|)-\mathbf{u}\frac{d\mathbf{z}}{ds}\right)ds+\frac{1}{2}\int_0^t\int_0^t\sum_{i,j=1}^{3}B_{ij}(\mathbf{z}(s_1)-\mathbf{z}(s_2))u_i(s_1)\right.$$
$$\left.\times u_j(s_2)ds_1ds_2:\quad \mathbf{z}(0)=\mathbf{0},|\mathbf{x}+\mathbf{z}(t)|=r,|\mathbf{x}+\mathbf{z}(s)|>r\right\}.$$

It should be noted that the Kolmogorov-Obukhov turbulence with $\sigma=0$ and $z=0$ corresponds to the ''frozen turbulence limit'' for which the large scale dynamics of reaction front is not as simple as for the case of ''rapid correlation time.'' It is easy to see that the line $\sigma=-3z+2/3$ represents the boundary between the different scaling regimes. Although the scaling functions (30) and (35) are the same at this line, namely, $\lambda(\epsilon)=\epsilon^{2/3-z}$, the functions $G(t,\mathbf{x})$ for both regions are discontinuous across their boundary. It would be interesting to investigate the source of this singular behavior as is done for turbulent diffusion [16,17]. It is also interesting to note that the boundary between these regions has shifted from what it would have been for turbulent diffusion: $\sigma+2z=0$. The reason for this phenomenon may be explained by the fact that even without convection term the appropriate scaling for the diffusion equation is $\lambda(\epsilon)=\epsilon^2$, but for the KPP equation it is $\lambda(\epsilon)=\epsilon$.

V. DISCUSSION AND CONCLUSIONS

We have presented an analytic study of the Kolmogorov-Petrovskii-Piskunov equation with convective term involving three-dimensional turbulent flow with arbitrarily many spatial and temporal scales. This has been made possible by the use of the exact renormalization theory and the Lagrangian stochastic differential equations that enable us to write a so-

lution of the KPP equation in terms of functional integrals.

We have developed an asymptotic method that gives the general formalism determining the upper bounds for the ensemble-averaged reaction front position and speed. We have found two distinct scaling regimes for which the phenomenon of infrared divergence plays the main role and molecular diffusion is unimportant. We have calculated the anomalous scaling functions and effective Hamiltonians for both regimes and showed that in the case of ''rapid correlation time,'' when renormalization procedure generates a white noise in time contribution, the upper bound can be obtained exactly.

In our analysis we have been restricted to the ensemble-averaged bounds on the reaction front. At present it is not clear whether the results on almost sure upper bounds obtained recently by Souganidis and Majda [13] can be extended to the three-dimensional random velocity field considered here. It was shown in [13] that for the stationary turbulent shear flow the bounds for the ensemble-averaged reaction speed can greatly overestimate almost every realization. It will be interesting to show the effect of such non-self-averaging behavior of reaction front for the realistic three-dimensional turbulent flow. It would be also interesting to consider the front propagation in R^d [25] and examine the problem of statistical universality considering non-Gaussian velocity statistics [17,26].

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