Calculus and Vectors B - MATH10131

Problem Sheet for Week 9 Integration & Vectors

Suggested reading: 'Stewart' Chapters 12 and 13

Easy Questions

1. Use vectors to find the three angles of each of the triangles with vertices at

   (a) $\vec{\mathbf{0}} = (0,0,0)$, $\vec{\mathbf{A}} = (0,0,5)$ and $\vec{\mathbf{B}} = (4,-3,0)$

   (b) $\vec{\mathbf{A}} = (0,1,-1)$, $\vec{\mathbf{B}} = (1,-1,0)$ and $\vec{\mathbf{C}} = (-1,0,1)$

   Note: if you can't find the angle without using a calculator, leave your answer in the form: $\cos^{-1}\cdot$.

   Hint: Note that in using $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$ to find the angle '*between*' the vectors, both vectors should either end at the same point or begin at the same point.

2. Find the areas of each of the triangles with vertices at

   (a) $\vec{\mathbf{A}} = (2,7,1)$, $\vec{\mathbf{B}} = (2,8,0)$ and $\vec{\mathbf{C}} = (3,6,1)$

   (b) $\vec{\mathbf{A}} = (1,-1,0)$, $\vec{\mathbf{B}} = (2,-2,1)$ and $\vec{\mathbf{C}} = (-3,0,4)$

   (c) $\vec{\mathbf{A}} = (0,1,0)$, $\vec{\mathbf{B}} = (-1,0,1)$ and $\vec{\mathbf{C}} = (2,3,-1)$

   Hint: Area of triangle ABC is $\frac{1}{2}\,\text{base}\times\text{height} = \frac{1}{2}|\vec{\mathbf{AB}}||\vec{\mathbf{AC}}|\sin\alpha = \frac{1}{2}|\vec{\mathbf{AB}}\times\vec{\mathbf{AC}}|$ where $\alpha = \angle\text{CAB}$

3. What are the volumes of the parallelepipeds with three of their edges given by the vectors

   (a)★ $\vec{\mathbf{a}} = (2,7,1)$, $\vec{\mathbf{b}} = (2,8,0)$, and $\vec{\mathbf{c}} = (3,5,0)$?

   (b) $\vec{\mathbf{a}} = (2,-2,1)$, $\vec{\mathbf{b}} = (-1,0,1)$, and $\vec{\mathbf{c}} = (0,0,5)$?

   (c) $\vec{\mathbf{a}} = (3,1,-4)$, $\vec{\mathbf{b}} = (0,1,0)$, and $\vec{\mathbf{c}} = (2,3,-1)$?

   Hint: Volume of the parallelepiped is $\left|\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})\right|$

4. If $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ find

   (a) $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ (b) $\left|\hat{\mathbf{j}}\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\right|$ (c) $\left|(\vec{\mathbf{a}}+\vec{\mathbf{b}})\times(\vec{\mathbf{a}}-\vec{\mathbf{b}})\right|$

Standard Questions

5. Distance ℓ of the point $\vec{\mathbf{P}}$ from the line $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\,\vec{\mathbf{v}}$ can be calculated as $\ell = \frac{|(\vec{\mathbf{P}}-\vec{\mathbf{r}}_0)\times\vec{\mathbf{v}}|}{|\vec{\mathbf{v}}|}$
   In each of the following examples, find how far is the point from the line

   ★(a) the point is $(1,0,0)$ and the line follows $x = 1-2t$, $y = 4t$ and $z = 2-3t$

   (b) the point is $(9,2,4)$ and the line passes through $(8,3,4)$ parallel to the line $\vec{\mathbf{r}} = (1,1,0) + t\,(1,0,-1)$

   (c) the point is where $x = 1-2t$, $y = 4t$, $z = 2-3t$ and $x+2y+z+5 = 0$ intersect and the line passes through the origin perpendicular to both $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$

   (d) the point is $(7,4,1)$ and the line passes through the two points $(6,4,1)$ and $(6,3,-1)$

(e) the point is $(1,1,1)$ and the line is where the two planes $x+y=1$ and $y=0$ intersect

6. Distance ℓ of the point $\vec{\mathbf{P}}$ from the plane $(\vec{\mathbf{r}}-\vec{\mathbf{r}}_0)\cdot\vec{\mathbf{n}}=0$ can be calculated as $\ell=\dfrac{|(\vec{\mathbf{P}}-\vec{\mathbf{r}}_0)\cdot\vec{\mathbf{n}}|}{|\vec{\mathbf{n}}|}$

   In each of the following examples find how far is the point from the plane

   ★(a) the point is $(1,4,-3)$ and the plane is the x-y plane (i.e. the plane containing the x and y axes)

   (b) the point is $(8,1,0)$ and the plane passes through the origin at right angles to $\sqrt{2}\,\widehat{\mathbf{j}}-\sqrt{2}\,\widehat{\mathbf{k}}$

   (c) the point is $(1,0,1)$ and the plane passes through the three points $(0,1,0)$, $(0,0,-1)$ and $(1,0,0)$

   (d) the point is $(0,1,0)$ and the plane passes through $(1,0,0)$ parallel to $x=0$

   (e) the point is $(0,0,0)$ and the plane passes both through $(1,2,3)$ and the line $\vec{\mathbf{r}}=(0,1,2)+t\,(3,1,-1)$