Calculus and Vectors B - MATH10131

Problem Sheet for Week 5 — Integration

***Suggested reading*:** 'Stewart' Chapters 5 and 7

Easy Questions

1. Provide formulae for the following indefinite integrals

   (a) $\int \sin\theta \, \mathrm{d}\theta$ (b) $\int \cos x \, \mathrm{d}x$ (c) $\int \sec^2 r \, \mathrm{d}r$

   (d) $\int s^{13} \, \mathrm{d}s$ (e) $\int \sqrt[7]{t} \, \mathrm{d}t$ (f) $\int \frac{1}{u} \, \mathrm{d}u$

   (g) $\int \frac{1}{|v|} \, \mathrm{d}v$ (h) $\int \sinh w \, \mathrm{d}w$ (i) $\int \cosh x \, \mathrm{d}x$

   Try to memorise all of these basic integrals.

2. Calculate the following definite integrals.

   (a) $\int_{-\pi/2}^{\pi} \sin\theta \, \mathrm{d}\theta$ (★b) $\int_{-\pi/2}^{\pi} \cos x \, \mathrm{d}x$ (c) $\int_0^1 s^{13} \, \mathrm{d}s$

   (★d) $\int_0^{128} \sqrt[7]{t} \, \mathrm{d}t$ (e) $\int_1^e \frac{1}{u} \, \mathrm{d}u$ (f) $\int_0^{\ln 2} \sinh w \, \mathrm{d}w$

   (g) $\int_0^{\ln 3} \cosh x \, \mathrm{d}x$ (h) $\int_0^{\ln 5} \mathrm{sech}^2 y \, \mathrm{d}y$ (i) $\int_{-1}^1 e^z \, \mathrm{d}z$

Standard Questions

3. Provide formulae for the following indefinite integrals

   (★a) $\int \frac{3}{2}\sin(4\theta - \pi) \, \mathrm{d}\theta$ (b) $\int \left(\sinh\frac{3-w}{2} + \cosh\frac{w-2}{3} + \mathrm{sech}^2\frac{w-1}{4} + e^{5w-1/5}\right) \mathrm{d}w$

   (c) $\int \left(3 + 7(4s - \tfrac{1}{3})^4\right) \mathrm{d}s$ (d) $\int \left(\frac{\pi}{\sqrt{(2-r)^2-1}} - \frac{2}{\sqrt{(r-3)^2+1}} + \frac{3}{\sqrt{1-(1-2r)^2}}\right) \mathrm{d}r$

4. Find the following integrals

   (★a) $\int \frac{3}{8 - 4x + x^2} \, \mathrm{d}x$ ( ) $\int \frac{3}{\sqrt{12x - 6 - 4x^2}} \, \mathrm{d}x$

Harder Questions

5. Show that the following recursion relations hold for the integrals given.

   (a) $K_n = nK_{n-1}$ for $n > 0$, where $K_n = \int_0^\infty x^n e^{-x} \, \mathrm{d}x$.

      i. by direct integration calculate K_0 and hence find the value of K_n for any $n \in \mathbb{N}$.
      ii. given that $K_{1/2} = \frac{1}{2}\sqrt{\pi}$ what is the value of $K_{7/2}$?

   (In fact, the Gamma function, defined by $\Gamma(r+1) = \int_0^\infty x^r e^{-x} \, \mathrm{d}x$ extends the factorial $r!$ to $r \notin \mathbb{N}$)

   (b) $I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n}\cos x \sin^{n-1} x$ where $I_n = \int \sin^n x \, \mathrm{d}x$. Why is this true only for $n \geq 2$?

i. by direct integration calculate I_0 and use the formula to find $\int \sin^6 x \, \mathrm{d}x$

ii. by direct integration calculate I_1 and use the formula to find $\int \sin^7 x \, \mathrm{d}x$

(c) $J_n = \dfrac{n-1}{n} J_{n-2} + \dfrac{1}{n} \sin x \cos^{n-1} x$ where $I_n = \displaystyle\int \cos^n x \, \mathrm{d}x$. Why is this true only for $n \geq 2$?

i. by direct integration calculate J_1 and use the formula to find $\int \cos^5 x \, \mathrm{d}x$

ii. by direct integration calculate J_0 and use the formula to find $\int \cos^4 x \, \mathrm{d}x$