Calculus and Vectors B - MATH10131

Problem Sheet for Week 4 — Derivatives, Series, Complex Numbers

***Suggested reading*:** 'Stewart' Chapters 3, 4, 11 and Appendix G

Easy Questions

1. Consider the function $f(x) = x - 5x^2$. Find
   (a) the derivative of $f(x)$ from the definition of $f'(x)$; (b) the equation of the tangent at $(1, -4)$

⋆2. Find all stationary (critical) points (if any) of the following functions and say whether each is a maximum, minimum or point of inflection
   (a) $\dfrac{1}{1+x}$ (b) $1 + x^4$ (c) $3x + x^3$ (d) $3x - x^3$

3. Find the absolute maximum and absolute minimum values of f on the given interval:
   (a) $f(x) = 3x^2 - 12x + 5, \quad [0, 3]$ (b) $f(x) = x^4 - 2x^2 + 3, \quad [-2, 3]$

4. The position of a car is given by $s(t) = 2 + 3t + \frac{1}{2}t^2 \quad t \geq 0$, where s is measured in metres.
   (a) Find the velocity after 2 s; (b) How long does it take for the velocity to reach 20 m/s

⋆5. Find the Maclaurin series for the functions
   (a) $\cos x$ (b) e^{5x} (c) $\sinh x$ Ans: $\sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$.

6. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ write the following complex numbers in the form $re^{i\theta}$. Sketch all of the points on a single diagram of the complex plane.
   (a) 1 (b) $-i$ (c) i (d) -1
   (e) $1 + i$ (f) $1 - i$ (g) $-1 + i$ (h) $-1 - i$
   (i) $1 + i\sqrt{3}$ (j) $\sqrt{3} - i$ (k) $-1 + i\sqrt{3}$ (l) $-\sqrt{3} - i$

Standard Questions

⋆7. Find the equation of the tangent line to the curve having equation (hint: use implicit differentiation)
   $x^2 + y^2 - 3xy + 4 = 0$ at the point $(2, 4)$ Ans: $y = 4x - 4$

⋆8. Sketch the function $\frac{8t}{4+t^2}$, making sure to locate all stationary (critical) points.

9. Find an equation of the tangent line to the hyperbola
   $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) Ans: $y = \frac{b^2 x_0}{a^2 y_0}(x - x_0) + y_0$

10 Use logarithmic differentiation to find the derivative of the function
   (a) $y = (2x+1)^5\left(x^4 - 3\right)^6$ (b) $y = x^{e^x}$ Ans: $y' = x^{e^x}e^x\left(\ln x + \frac{1}{x}\right)$ (c) $y = x^x$

11. (a) Show that any function of the form
   $y = A\sinh(mx) + B\cosh(mx)$ satisfies the differential equation $\frac{d^2y}{dx^2} = m^2 y$.
   (b) Find $y = y(x)$ such that $\frac{d^2y}{dx^2} = 9y, \quad y(0) = -4$, and $\frac{dy}{dx}(0) = 6$.
   Ans: $y = 2\sinh(3x) - 4\cosh(3x)$

12. Find the Taylor series for the function f at the given value of a.

(a) $\cos x \qquad a = \pi$ (b) $1 + x + x^2 \qquad a = 2$

Ans: (a) $\cos x = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4 + ... = \sum_0^\infty \frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$

Harder Questions

13. Use Euler's formula to obtain the identity $\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$. Find a corresponding identity for $\sin(4\theta)$?