Calculus and Vectors B - MATH10131

Problem Sheet 3 — Limits and Derivatives

***Reminder*:**

- attempt all 'easy' questions without delay and discuss any difficulties in your Supervision Class
- attempt the starred questions and hand in your answers (complete or incomplete) to your supervisor
- the 'harder' questions are optional, offering an additional challenge as well as introducing problems of further interest

***Suggested reading*:** 'Stewart' chapters 2, 3 and 4

Easy Questions

1. Differentiate the following functions. You should be able to write down the answers at once without any intermediate working. If necessary, practice on examples from the textbook until you can do this with confidence.

   (a) $(1-x^2)^3$ (b) $\sin^2 t$ (c) $\dfrac{1}{1-x}$ (d) $\theta^2 \tan\theta$

   (e) $\sqrt{u^3+u}$ (f) $e^{-t}\cosh t \sinh t$ (g) $x^3 \exp(-2x)$ (h) $z^{-2}\ln(1+z^2)$

⋆2. Find the following limits

   (a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x}$ (b) $\lim\limits_{x\to -\infty} \dfrac{\ln(1+e^x)}{x}$ (c) $\lim\limits_{x\to 0} \dfrac{\sinh(x)}{x}$

3. Find a value A (if it exists) for which the following functions will be continuous for all $x \in \mathbb{R}$

   (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$ (b) $g(x) = \begin{cases} 1/\sqrt{x^4} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$

Standard Questions

4. Find the following derivatives. You should work efficiently, writing down as few intermediate steps as possible.

   (⋆a) $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin^2 x \cos^3 x$ (⋆b) $\dfrac{\mathrm{d}}{\mathrm{d}u}(au+b)^{3/2}$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}a}(au+b)^{3/2}$

   (⋆d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{\dfrac{a+x}{a-x}}$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}p} p\sin(s/p)$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\theta^n \sin^m(a\theta+b)$

5. Find a general formula for the n^{th} derivative of (a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1-x}$

⋆6. Find a value A (if it exists) for which the following function will be continuous for $t \geq 0$

$$g(t) = \begin{cases} t^t & \text{if } t > 0 \\ A & \text{if } t = 0 \end{cases}$$

Hint. You can write t^t as $e^{t \ln t}$.

7. Are the following true or false?

(a) $\lim_{x \to 0} \dfrac{1}{1 + e^{1/x}} = 0$ (b) $\lim_{x \to 1} \tan^{-1} \dfrac{1}{1 - x} = \dfrac{\pi}{2}$

8. Where are the functions $f(g(x))$ and $g(f(x))$ continuous or discontinuous if

$$f(x) = 1 + x^2 \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$$

Harder Questions

9. The function $[\,\cdot\,]$ represents the 'integer part' of any number, defined such that $[x]$ is the highest integer value that is less than or equal to x. Thus $[\pi] = 3$, $[e] = 2$, $[5] = 5$, $[-4.9] = -5$, etc.

Where are the following functions discontinuous, if at all

(a) $x - [x]$ (b) $[\sin x]$ (c) $[1/x]$ (d) $(-1)^{[x]}$ (e) $(-1)^{2[\ln x^2]}$