Problem Sheet 10 $\boxed{\textit{MATH10131}}$ **Partial Derivatives**

Suggested reading: 'Stewart' Chapter 14

Easy Questions

1. Give *all* first and second-order partial derivatives of the following functions:
   (a) $f(p,q) = \sin(p-q)$ (b) $g(r,s) = r^2 + 2rs - s^2$
   (c) $h(t,u) = e^{t-u^2}$ (d) $u(v,w) = v^2/w + \ln(3v + 2vw)$
   (e) $v(x,y) = x^y$

2. Find $\mathrm{d}z/\mathrm{d}t$ in each of the following cases, using two methods: i. by using the chain-rule for partial derivatives; and ii. by writing z in terms of t (i.e., eliminating x and y) and then differentiating
   (a) $z = 3x^2y^3$ with $x = t^4,\ y = t^2$ (b) $z = \exp(1-xy)$ with $x = t^{1/3},\ y = t^3$
   (c) $z = \sin(x+y)$ with $x = t^{-2},\ y = t^2$ (d) $z = \cosh(y+x)$ with $x = \ln t,\ y = 2/t$
   Both methods should give the same result in each case.

Standard Questions

3. (a) If $f = e^{-t}\cos x$ show that $f_t - f_{xx} = 0$ ('the heat equation')
   (b) If $f = e^{x-t} + e^{x+t}$ show that $f_{tt} - f_{xx} = 0$ ('the wave equation')
   (c) If $f = e^x(\cos y - \sin z)$ show that $f_{xx} + f_{yy} + f_{zz} = 0$ ('Laplace's equation' in 3 dimensions)

4. (a)★ If $y = \sin u$ and $u = r^2 + s^2$, find the partial derivatives $\left.\frac{\partial y}{\partial r}\right|_s$ and $\left.\frac{\partial y}{\partial s}\right|_r$
   (b) If $z = \sin(y^2x)$, $x = u^2 - v^2$ and $y = uv$, find the partial derivatives $\left.\frac{\partial z}{\partial u}\right|_v$ and $\left.\frac{\partial z}{\partial v}\right|_u$
   (c) If $w = \cos(t^2)$ and $t = xe^{-y}$, find the partial derivatives $\left.\frac{\partial w}{\partial x}\right|_y$ and $\left.\frac{\partial w}{\partial y}\right|_x$
   (d)★ If $x = \sin u - e^v$, $u = t^2 + s^2$ and $v = t/s$, find the partial derivatives $\left.\frac{\partial x}{\partial t}\right|_s$ and $\left.\frac{\partial x}{\partial s}\right|_t$

5. Find all critical points of the following functions and identify whether each one is a maximum, a minimum or a saddle point.
   ★(a) $f(x,y) = 6xy + x^3 - y^2$
   (b) $f(x,y) = y^2 + x^2 - 2x$
   (c) $f(x,y) = x^5 + y^4 - 5x - 32y - 3$
   (d) $f(x,y) = x^2 - 4x + 2xy - y^2 + 5$

6. Find the Taylor series expansions (only quadratic approximation) of the following functions of x and y about the points given
   ★(a) $f(x,y) = 2x^2 - xy - y^2 - 6x - 3y + 5$ about $(x,y) = (1,-2)$
   (b) $f(x,y) = e^{-x^2-y^2}$ about $(x,y) = (0,0)$