Superdiffusion in self-reinforcing run-and-tumble model with rests

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This paper introduces a run-and-tumble model with self-reinforcing directionality and rests. We derive a single governing hyperbolic partial differential equation for the probability density of random-walk position, from which we obtain the second moment in the long-time limit. We find the criteria for the transition between superdiffusion and diffusion caused by the addition of a rest state. The emergence of superdiffusion depends on both the parameter representing the strength of self-reinforcement and the ratio between mean running and resting times. The mean running time must be at least $2/3$ of the mean resting time for superdiffusion to be possible. Monte Carlo simulations validate this theoretical result. This work demonstrates the possibility of extending the telegrapher's (or Cattaneo) equation by adding self-reinforcing directionality so that superdiffusion occurs even when rests are introduced.

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I. INTRODUCTION

Persistent random walks with finite velocities are powerful models describing chemotaxis [1–5], organism movement and searching strategies [6–8], intracellular transport [9–11], and cell motility [12,13]. Stochastic cell movement plays a major role in embryonic morphogenesis, wound healing, and tumor cell proliferation [14]. The modeling of cell and bacteria migration toward a favorable environment is usually based on "velocity-jump" models describing self-propelled motion with the runs and tumbles. Finite velocities and inertial resistance to changes in direction make these random walks physically well motivated since random walkers in nature cannot instantaneously jump to different states. The collective behavior of cells and various organisms is another rapidly growing area of active matter research [15,16]. Various hyperbolic models involving nonlinear partial differential equations (PDEs) for the population densities have been used for analysis of spatiotemporal patterns describing the chemical and social interactions of organisms [17–21].

Models of cell motility have been predominantly concerned with Markovian random-walk models (see for example Refs. [13,22]). However, the analysis of random movement of metastatic cancer cells shows the anomalous superdiffusive dynamics of cell migration [23]. Over the past few years there have been several attempts to model anomalous transport involving superdiffusion [24–30]. Superdiffusion occurs as a result of the power-law distributed running times with infinite second moment [25] or collective interaction between random walkers [31]. Such models are intrinsically non-Markovian involving nonlocal in time integral terms, making the inclusion of reactions, internal dynamics, chemical signals, and interparticle interactions cumbersome and unwieldy.

Recently, we introduced a persistent random-walk model with self-reinforcing directionality that generates superdiffusion from exponentially distributed runs, accurately modeling the statistics found in active intracellular transport [32]. Although this model involves strong memory, it can be formulated as a persistent random walk with space- and time-dependent coefficients, facilitating convenient implementations of reactions, chemotaxis, and interactions using the established methods within the persistent random-walk framework.

In Ref. [32], we considered a particle moving with velocity $\pm\nu$ for exponentially distributed running times with rate λ. The key idea was to introduce conditional transition probabilities, q_+ and q_-, involving self-reinforcing directionality. These conditional transition probabilities describe switching from one velocity state to the other dependent on the time that the particle has spent in the respective states such that $q_\pm = wt^\pm/t + (1-w)t^\mp/t$. In this case, t^+ and t^- are the times that a particle has spent traveling in the positive and negative direction, respectively, and $t = t^+ + t^-$. The persistence probability w defines how much the random walk chooses to follow its past behavior. For example, if much time is spent moving in the positive direction ($t^+ \to t$) and $w = 1$, then, the particle will choose to move in the positive direction with probability $q_+ \to 1$. To formulate the governing equations, we introduce $p_+(x,t)$ and $p_-(x,t)$, which are the joint probability densities that the position of the particle is in the interval $(x, x+dx)$ at time t and moving with positive and negative

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velocities, respectively. Then

$$\frac{\partial p_\pm}{\partial t} \pm \nu \frac{\partial p_\pm}{\partial x} = -\lambda(1-q_\pm)p_\pm + \lambda(1-q_\mp)p_\mp. \quad (1)$$

The advantage of this formulation is that $q_\pm$ can be simply expressed as a function of space, x, and time, t. If one realizes that $x = \nu(t_+ - t_-)$, then

$$q_\pm(x,t) = \frac{1}{2}\left[1 \pm \frac{(2w-1)(x-x_0)}{\nu t}\right]. \quad (2)$$

Expressing $q_\pm$ in this way, we can write down (1) as a single hyperbolic PDE,

$$\frac{\partial^2 p}{\partial t^2} + \lambda\frac{\partial p}{\partial t} = \nu^2\frac{\partial^2 p}{\partial x^2} - \frac{\lambda(2w-1)}{t}\frac{\partial[(x-x_0)p]}{\partial x}. \quad (3)$$

This model has been shown to exhibit superdiffusion despite having exponentially distributed run times [32]. For values of $w > 1/2$, the conditional transition probabilities generates self-reinforcing directionality in (3) and for $w > 3/4$, superdiffusion. Research on reinforcement in random walks has been explored in jump processes [22]. The model represented in (3) is actually a continuous space and time generalization of the elephant random walk [33–39], which is discrete in space and time.

A limitation of (3) is that only active states were included in the model. In reality, most natural phenomena have rest states associated with passive movement, no movement, or even death. In particular, animals move by alternating between foraging and resting [40,41]. In modeling processes with rest states, the Lévy walk with rests [42–45] and persistent random walks with death [46] have been introduced.

The aim of this paper is to formulate the self-reinforcing velocity random walks with stochastic rests. Important questions for this model are does superdiffusion still exist after introducing rests? and; if superdiffusion does exist, then what is the critical value of the ratio of mean running and resting time for which the phase transition from diffusion to superdiffusion occurs?

In the first section, we formulate the self-reinforcing directionality random walk with a rest state and derive the nonlocal hyperbolic governing partial differential equation for the PDF of particle position. In the second section, we find an analytical expression for the second moment and the critical point where the transition from diffusion to superdiffusion occurs. Finally, we present the Monte Carlo simulations of the random walk with reinforcement, which confirms the existence of superdiffusion.

II. SELF-REINFORCING DIRECTIONALITY WITH RESTS

In this section, we introduce the self-reinforcing velocity random walk with transitions between moving states via an intermediate resting state with zero velocity. Consider a particle that moves with constant speed ν in the positive or negative direction for exponentially distributed running times with rate λ. This movement is interrupted by rests with exponentially distributed resting times with rate η. Now we introduce three joint probability density functions, $p_+(x,t)$, $p_-(x,t)$, and $p_0(x,t)$. Here $p_+(x,t)$ and $p_-(x,t)$ are the same as the joint densities described in (1). Additionally, $p_0(x,t)$ is the joint

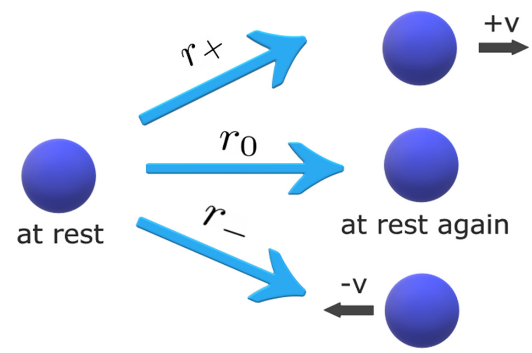

FIG. 1. A diagram showing the conditional transition probabilities, r_+, r_-, and r_0, for the velocity random walk in (4). A particle at rest can switch to the positive velocity state, negative velocity state, or remain at rest.

probability density that a particle is in the interval $(x, x+dx)$ at time t and has zero velocity. The governing equations for these probability densities are

$$\frac{\partial p_\pm}{\partial t} \pm \nu\frac{\partial p_\pm}{\partial x} = -\lambda p_\pm + \eta r_\pm p_0,$$
$$\frac{\partial p_0}{\partial t} = \lambda p_+ + \lambda p_- - \eta(1-r_0)p_0. \quad (4)$$

Here the transition probabilities, r_+, r_-, and r_0, describe three possible transitions that the particle can make from the rest state. r_+ is the probability that the particle switches from the rest state to the moving state with positive velocity, ν. r_- is the probability of switching from the rest state to the moving state with negative velocity $-\nu$. r_0 is the probability that the resting particle remains at rest again after an exponentially distributed random time with rate η (see Fig. 1). Clearly, $r_+ + r_- + r_0 = 1$.

In this paper, we introduce self-reinforcing directionality through the conditional transition probabilities as follows:

$$r_\pm = w_1\frac{t^\pm}{t} + w_2\frac{t^\mp}{t} + w_3\frac{t^0}{t}, \quad (5)$$

where t^+, t^-, and t^0 are the relative times that the particle has spent in the positive velocity, negative velocity, or resting state, respectively. The total time is $t = t_+ + t_- + t_0$. The weights, w_1, w_2, and w_3, represent the amount of influence that each relative time has on the probability that a particle will transition to the corresponding state. Naturally, the weights are positive and $w_1 + w_2 + w_3 = 1$.

Why and how does (5) introduce self-reinforcing directionality into (4)? We demonstrate the effect on the conditional transition probabilities by considering weights w_1 and w_2. For $w_1 > w_2$, the random walk reinforces its own past behavior by increasing the transition probability to the positive velocity state, r_+, when the time spent in that state, t^+, increases. The same can be said between r_- and t^-. In other words, the more the random walk spends time in either the positive or negative velocity state, the more likely a future transition into that state becomes. So the weights w_1 and w_2 perform an essential function in self-reinforcing directionality by either "punishing" or "rewarding" past choices and making future transitions to states dependent on time spent in the two active states. Now we present a clear and effective method for simplifying (5) so that a single governing equation can be obtained.

We can rewrite (5) using $t = t^+ + t^- + t^0$ and $x = x_0 + v(t^+ - t^-)$ as

$$r_+(x,t) = \frac{w_1 - w_2}{2}\frac{x - x_0}{vt} + \frac{w_1 + w_2}{2} + \Lambda\frac{t^0}{t}$$
$$r_-(x,t) = -\frac{w_1 - w_2}{2}\frac{x - x_0}{vt} + \frac{w_1 + w_2}{2} + \Lambda\frac{t^0}{t} \quad (6)$$
$$r_0 = 1 - r_+ - r_-,$$

where $\Lambda = -(w_1 + w_2)/2 + w_3$.

In this paper, we introduce self-reinforcing directionality such that t_0, the time spent resting, does not explicitly contribute to the conditional transition probabilities r_+ and r_-. To achieve this, we set $\Lambda = 0$ and given $w_1 + w_2 + w_3 = 1$, one finds that $w_3 = 1/3$ and $w_1 + w_2 = 2/3$ is a unique requirement. Self-reinforcement appears when $w_1 > w_2$ and disappears for the symmetrical case when $w_1 = w_2 = w_3 = 1/3$. Then the conditional transition probabilities in (6) can be written in terms of a self-reinforcing parameter, α_0, as

$$r_\pm = \frac{1}{3} \pm \alpha_0\frac{x - x_0}{2vt} \quad \text{and} \quad r_0 = \frac{1}{3}, \quad (7)$$

where

$$\alpha_0 = w_1 - w_2 \quad \text{and} \quad 0 < \alpha_0 < 2/3. \quad (8)$$

The formulation of self-reinforcement in this way presents a particularly powerful mechanism to introduce memory effects and superdiffusion. It is clear that this mechanism is different to that used to generate superdiffusion in continuous time random walks or Lévy walks. Note that (7) is also valid for $-2/3 < \alpha_0 \leqslant 0$, for which the model exhibits behavior opposite to self-reinforcement. Using the definition of conditional transition probabilities in (7), we can formulate a single governing equation that enables various extensions, such as reactions, interactions, and chemotaxis, to be readily applied from the persistent random-walk framework. In our previous paper, we suggested a simple microscopic mechanism of self-reinforcement (see Sec. VII in Ref. [32]).

Now we will derive the single governing equation. From combining (4), we obtain three equations,

$$\frac{\partial p}{\partial t} = -\frac{\partial J}{\partial x}, \quad \frac{\partial p_0}{\partial t} = \lambda p - \gamma p_0, \quad \text{and}$$
$$\frac{\partial J}{\partial t} = -v^2\frac{\partial p}{\partial x} + v^2\frac{\partial p_0}{\partial x} - \lambda J + v\eta(r_+ - r_-)p_0, \quad (9)$$

where $p(x,t) = p_+(x,t) + p_-(x,t) + p_0(x,t)$ so that $p(x,t)$ is the probability density of finding the particle in the interval $(x, x + dx)$ at time t regardless of the particle's velocity state. Furthermore, $\int_{-\infty}^{\infty} p(x,t)dx = 1$. In addition, $J = vp_+ - vp_-$ and

$$\gamma = \lambda + (1 - r_0)\eta = \lambda + \tfrac{2}{3}\eta. \quad (10)$$

The initial conditions are

$$p(x,0) = \delta(x - x_0), \quad p_0(x,0) = 0 \quad \text{and}$$
$$J(x,0) = v(2u - 1)\delta(x - x_0), \quad (11)$$

where u is the probability that the particle begins with positive velocity and $(1 - u)$ to begin with negative velocity. Solving the second equation in (9) with the initial condition $p_0(x,0) = 0$, one can also write $p_0(x,t)$ in terms of $p(x,t)$ as

$$p_0(x,t) = \lambda\int_0^t e^{-\gamma(t-t')}p(x,t')dt'. \quad (12)$$

Combining (9), (12), and (7), a single equation can be found for p as

$$\frac{\partial^2 p}{\partial t^2} + \lambda\frac{\partial p}{\partial t} - v^2\frac{\partial^2 p}{\partial x^2} + \lambda v^2\int_0^t e^{-\gamma(t-t')}\frac{\partial^2 p(x,t')}{\partial x^2}dt'$$
$$+\frac{\lambda\alpha_0\eta}{t}\frac{\partial}{\partial x}\left[(x - x_0)\int_0^t e^{-\gamma(t-t')}p(x,t')dt'\right] = 0. \quad (13)$$

Now the crucial question is does the intermediate rest state destroy superdiffusion seen in the self-reinforcing directionality random-walk model? To answer this, we perform moment analysis.

If the parameter $\eta \to \infty$, then the average rest time, which is $1/\eta$, approaches 0. The fourth term in (13) approaches 0 because γ defined in (10) $\to \infty$. However, the last term in (13) does not approach 0 because $(1 - r_0)\eta e^{-\gamma(t-t')} \to \delta(t - t')$ as $\eta \to \infty$. So in this case, (13) becomes the same as the governing equation in the case of no rests, which can be found in (10) in Ref. [32].

III. MOMENT CALCULATIONS AND SUPERDIFFUSION

To find an analytical expression for the second moment $\mu_2(t) = \int_{-\infty}^{\infty} x^2 p(x,t)dx$, we use (13) with the assumption that $x_0 = 0$. Then

$$\frac{d^2\mu_2(t)}{dt^2} + \lambda\frac{d\mu_2(t)}{dt} - \frac{2\lambda\alpha_0\eta}{t}\int_0^t e^{-\gamma(t-t')}\mu_2(t')dt'$$
$$= 2v^2\left(1 - \frac{\lambda}{\gamma}\right) + \frac{2v^2\lambda}{\gamma}e^{-\gamma t}. \quad (14)$$

Now using the initial conditions (11), we obtain the initial conditions for the second moment,

$$\mu_2(0) = 0 \quad \text{and} \quad \frac{d\mu_2(0)}{dt} = 0. \quad (15)$$

Using the Laplace transform of (14) and (15), the equation for $\hat{\mu}_2(s) = \int_0^\infty \mu_2(t)e^{-st}dt$ is

$$\frac{d\hat{\mu}_2}{ds} + \frac{2s + \lambda + 2\lambda\alpha_0\eta(s + \gamma)^{-1}}{s(s + \lambda)}\hat{\mu}_2$$
$$= -\frac{2v^2}{s^3(s + \lambda)}\left(1 - \frac{\lambda}{\gamma}\right) - \frac{2v^2\lambda}{\gamma s(s + \gamma)^2(s + \lambda)}. \quad (16)$$

Now let us look at the long-time limit ($s \to 0$) for (16), then

$$\frac{d\hat{\mu}_2}{ds} + \frac{1 + \frac{2\alpha_0\eta}{\gamma}}{s}\hat{\mu}_2 \approx -\frac{2v^2}{s^3\lambda}\left(1 - \frac{\lambda}{\gamma}\right) - \frac{2v^2}{s\gamma^3}. \quad (17)$$

When neglecting the rest state, $\eta \to \infty$ or $\gamma = \lambda + (1 - r_0)\eta \to \infty$, then (17) becomes

$$\frac{d\hat{\mu}_2}{ds} + \frac{1 + 2\alpha_0}{s}\hat{\mu}_2 \approx -\frac{2v^2}{s^3\lambda}. \quad (18)$$

The homogeneous solution for (18) is $\hat{\mu}_2(s) = C_2 s^{-2\alpha_0 - 1}$ where C_2 is a constant, which gives $\mu_2(t) \sim t^{2\alpha_0}$ taking the

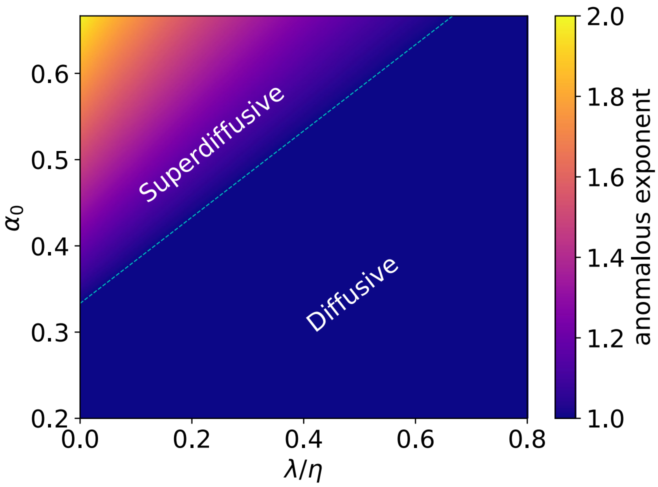

FIG. 2. A diagram showing where the diffusive and superdiffusive regimes are found for varying values of α_0 and λ/η. The cyan dashed line shows $\alpha_0 = \lambda/2\eta + 1/3$. The anomalous exponent is defined in (25).

inverse. This shows that $2\alpha_0$ is the anomalous exponent. Analogously, from (17) we obtain

$$\hat{\mu}_2(s) \sim C_2 s^{-\frac{2\alpha_0\eta}{\gamma}-1}, \tag{19}$$

which gives

$$\mu_2(t) \sim C_2 t^{\frac{2\alpha_0\eta}{\gamma}}. \tag{20}$$

This shows that even with rests, self-reinforcing directionality is enough to generate superdiffusion (see Fig. 2). The first moment, $\mu_1(t) = \int_{-\infty}^{\infty} x p(x,t) dx$, can be found in a similar way as

$$\mu_1(t) \sim C_1 t^{\frac{\alpha_0\eta}{\gamma}}, \tag{21}$$

where C_1 is a constant.

In the following sections, we confirm superdiffusion through Monte Carlo simulations for both the second moment and the variance $\mathrm{Var}[x(t)] = \mu_2(t) - [\mu_1(t)]^2$ (see Figs. 3 and 4). The Monte Carlo simulation results in Figs. 4 show that $C_1 \neq C_2$ (and further that $C_2 > C_1^2$) such that the variance is nonzero and follows the same time dependence as the second moment. Now let us consider for what parameter values superdiffusion is achieved.

For superdiffusion, the anomalous exponent in (20) must satisfy the condition

$$1 < \frac{2\alpha_0}{\frac{\lambda}{\eta} + \frac{2}{3}} < 2, \tag{22}$$

where (10) has been used to simplify the expression. Evidently, superdiffusion only depends on two parameters: the self-reinforcement parameter, α_0, and the ratio between run and rest rates, λ/η. Rearranging, (22) becomes

$$\frac{1}{3} + \frac{1}{2}\frac{\lambda}{\eta} < \alpha_0 < \frac{2}{3} + \frac{\lambda}{\eta}. \tag{23}$$

The left inequality gives $1/3 < \alpha_0$, which in conjunction with (8) means that $1/3 < \alpha_0 < 2/3$ is needed for superdiffusion.

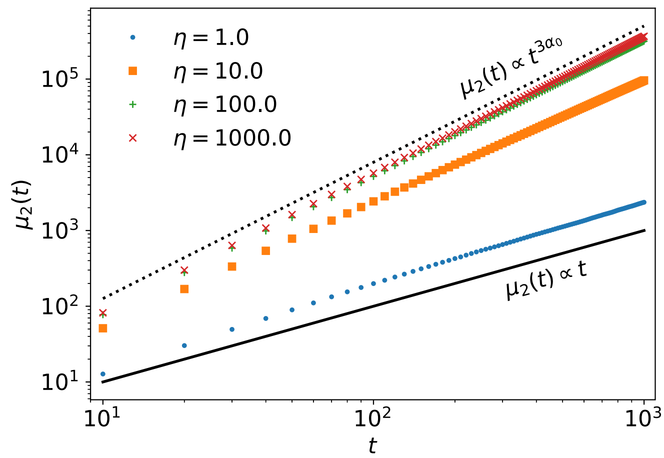

FIG. 3. Mean-squared displacements for the random-walk simulation with varying η. The parameters for the simulation were $\alpha_0 = 0.6 < 2/3$, $r_0 = 1/3$, $\lambda = 1$, $\nu = 1$ and the number of particles $N = 10^4$. The solid black line shows diffusion $\mu_2(t) \sim t$ and dashed black line shows the predicted superdiffusion from (20) and (22): $\mu_2(t) \sim t^{3\alpha_0}$ as $\lambda/\eta \to 0$ $(\eta \to \infty)$.

Then considering (8) again, we find the limits $0 \leqslant \lambda/\eta < 2/3$ are necessary for superdiffusion.

It follows from (20) that in the superdiffusive regime, the second moment

$$\mu_2(t) \sim t^{\sigma}, \tag{24}$$

where the anomalous diffusion exponent [17,44,47] is

$$\sigma = \frac{3\alpha_0}{1 + \frac{3}{2}\frac{\lambda}{\eta}} \tag{25}$$

with the bounds $1/3 < \alpha_0 < 2/3$ and $0 \leqslant \lambda/\eta < 2/3$. The phase diagram showing different parameter values and the corresponding superdiffusive or diffusive states can be seen in Fig. 2.

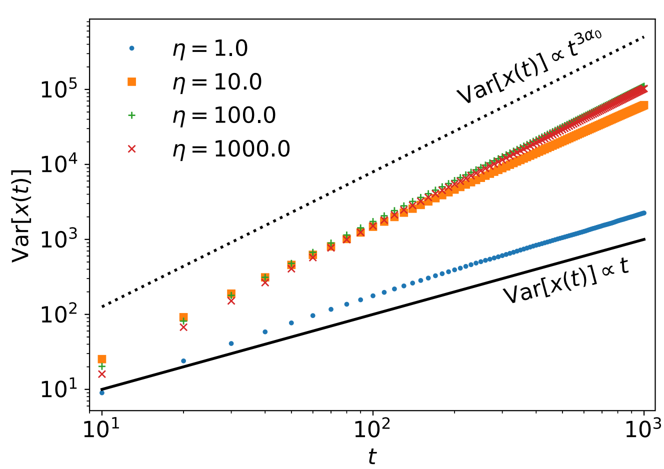

FIG. 4. The variance for the random-walk simulation with varying η. The parameters for this simulation were exactly the same as in Fig. 3. The solid and dashed black lines are also exactly the same, showing a constant multiplicative difference between the second moment and the variance

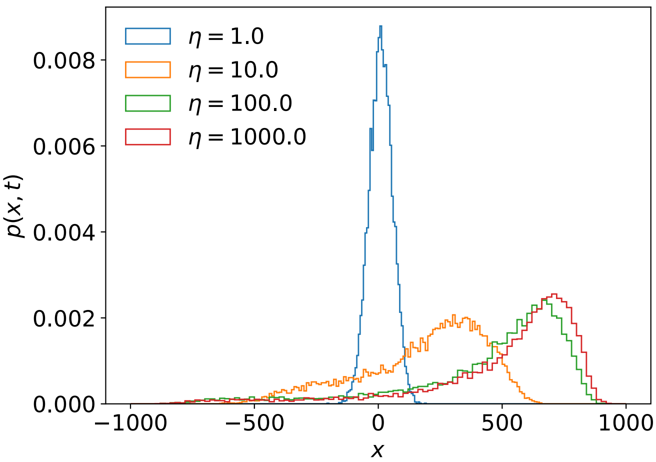

FIG. 5. PDF of particle positions at $t = 1000$ for the random-walk simulation with varying η. Identical simulation data from Fig. 3 was used. The parameters for the simulation were $\alpha_0 = 0.6$, $r_0 = 1/3$, $\lambda = 1$, and $\nu = 1$ and the number of particles $N = 10^4$.

It is particularly interesting to note that there is a smooth transition from diffusion to superdiffusion dependent on the ratio between running and resting rates, λ/η, in addition to the self-reinforcing parameter, α_0. In modeling various different transport phenomena with this self-reinforcing random walk with rests, we expect the dependence of the diffusion-superdiffusion transition on λ/η to be especially useful as there is a clear physical meaning to why superdiffusion emerges from a random walk with rests. For example, modeling transport mediated by multiple types of motor proteins will involve heterogeneous values of λ and η and may elucidate why some motor protein transport is more superdiffusive than others.

IV. MONTE CARLO SIMULATIONS

In this section, we validate the theoretical result in (20) and show the displacement PDFs as we vary η. The numerical simulations for a single random walk corresponding to Eq. (4) were performed as follows:

(1) Initialize variables for current simulation time $T_c = 0$, particle position $X_c = 0$ and current particle state $S_c = 1$. In this case, there are only three possible values for $S_c = 0$ or ± 1 corresponding to the rest, positive velocity and negative velocity states, respectively. For simplicity, we assume the random walk starts in the positive velocity state.

(2) Initialize the constants of the simulation: λ, η, ν, α_0, r_0, and t_{end}, the end time of simulation.

(3) If $S_c = 0$, then generate a random number $\Delta T = -\ln(U)/\eta$, where $U \in [0, 1)$ is a uniformly distributed random number. If $S_c = \pm 1$, then generate a random number $\Delta T = -\ln(U)/\lambda$. We emphasize that ΔT has exponential distribution with the density $\frac{d}{dt}\text{Prob}[\Delta T < t] = \eta \exp(-\eta t)$ for the rest state or $\lambda \exp(-\lambda t)$ for the moving states.

(4) Increment the current simulation time $T_c = T_c + \Delta T$ and the particle position $X_c = X_c + \nu S_c \Delta T$.

(5) If $S_c = \pm 1$, then set $S_c = 0$. If $S_c = 0$, then generate a uniformly distributed random number, $V \in [0, 1)$ and calculate $R_\pm = r_0 \pm \alpha_0 X_c/(2\nu T_c)$. For $0 \leqslant V < R_+$, set $S_c = 1$. For $R_+ \leqslant V < R_+ + R_-$ set $S_c = -1$. Otherwise, set. $S_c = 0$.

(6) Iterate steps 3 to 5 until $T_c \geqslant t_{\text{end}}$.

The numerical simulations in this paper were performed using Python3, taking advantage of the "Numba" package for JIT compilation and the "multiprocessing" package for CPU parallelization. These packages were used to significantly improve simulation execution times.

Figures 3 and 4 show the emergence of superdiffusion and excellent correspondence with (20). Figure 5 shows the behavior of the PDF as the value of η is varied. Clearly, when the rests become negligible in the asymptotic limit $\lambda/\eta \to 0$ ($\eta \to \infty$), the drift of particles caused by self-reinforced directionality dominates. This clearly shows that particles engage in self-reinforcing directionality as rest states become less time-consuming and particles choose to move in the same direction as their past history.

V. SUMMARY AND CONCLUSIONS

In this paper, we have formulated a run-and-tumble model with self-reinforcing directionality and rests. The system of PDEs (4) has been reduced to a single, nonlocal equation for the total probability density (13). From this single governing equation, we demonstrated the emergence of superdiffusion by deriving the second moment for the long-time limit. This emergence depends on two parameters: the self-reinforcement of particles, α_0, and the ratio between running and resting rates, λ/η. We find that at the critical point, $\lambda/\eta = 2/3$, superdiffusion emerges and remains for $\lambda/\eta < 2/3$. In other words, the mean running time must be at least 2/3 of the mean resting time for superdiffusion to occur in this model. Interestingly, we find that even a rest state cannot completely destroy the superdiffusion generated by self-reinforcement. Further, we present the method for Monte Carlo simulation of these random walks and show that the second moment corresponds with theoretical predictions. This superdiffusive model involving rests has potential application modeling the trapping of intracellular vesicles in actin-rich regions of neurons. This resting behavior is thought to act as functional reservoirs and help maintain the flow of presynaptic vesicles in the neurons of *Caenorhabditis elegans* [48].

Since our model describes an anomalous random walk with strong memory, it would be interesting to explore its ergodic properties considering both the Khinchin and fluctuation-dissipation theorems [47,49–52]. We expect ergodicity breaking for our model as is the case for the discrete random walk with global memory [53,54]. A natural extension of resting times distributed with constant rate η is to introduce a rest state that is non-Markovian with a residence time-dependent rate [55,56]. Recently, we considered the case with Mittag-Leffler distributed rest times for which the mean residence time in the rest state was divergent [57]. This dominates self-reinforcing directionality in the long-time limit and generates subdiffusion. It is also interesting to consider the case when the velocities alternate at nonexponentially distributed random times [58] or driven by random trials [59]. Furthermore, this new framework opens new avenues to include interactions of particles by density-dependent rates, $\lambda(p)$ and $\eta(p)$, and velocity, $\nu(p)$, leading to aggregation and pattern formation in active matter [16].

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