6 Options on assets paying dividends

6.1 Introduction

The majority of companies who have issued shares pay out dividends of some form another, fortunately it is relatively easy to incorporate dividend payments into the option pricing methodology. Of even greater use is that the methods used for pricing options on dividend paying stocks can be routinely extended to deal with other, analogous, problems such as options on foreign currency where the dividend becomes the foreign risk-free interest rate and options on commodities where the dividend becomes minus the cost of carry.

There are two main ways of modelling dividend payments: as continuous and as discrete.

6.2 Continuous constant dividend yield

This is the simplest payment structure, assume that over a period of time dt the underlying asset pays out a dividend $DSdt$ in that D is the proportion of the value of the asset paid out over this period of time. D is considered to be constant and independent of t though the size of the dividend will obviously depend on S which is dependent on t.

How does this affect our model? By using arbitrage arguments (see Examples 1) a payment of dividends results in the underlying asset price dropping by the value of the dividend. Hence with a continuous dividend the stochastic process is given by

$$dS = (\mu - D)Sdt + \sigma SdW.$$

To derive the governing PDE a similar process is followed but although the portfolio is still

$$\Pi = V - \Delta S$$

in this case

$$d\Pi = dV - \Delta(dS + DSdt)$$

as the holder of the portfolio receives the dividend as well. Proceeding as for the non-dividend case

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt,$$

and so

$$d\Pi = \frac{\partial V}{\partial t}dt + (\frac{\partial V}{\partial S} - \Delta)\left[(\mu - D)Sdt + \sigma SdW\right] - \Delta DSdt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt.$$

Setting

$$\Delta = \frac{\partial V}{\partial S}$$

leads to a deterministic result, to which we can apply the usual no-arbitrage argument, i.e.

$$\begin{aligned} d\Pi &= \frac{\partial V}{\partial t}dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - DS\frac{\partial V}{\partial S}dt \\ &= r\Pi dt \\ &= r(V - S\frac{\partial V}{\partial S})dt \end{aligned}$$

which gives the following PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0. \tag{38}$$

The standard Black-Scholes equation derived earlier in the course is just a special case of this equation for the case when $D = 0$. Valuing European call and put options is reasonably straightforward, the main difference being that r is replaced by $r - D$ but only in the coefficient of the $\partial C/\partial S$. To account for this slight difference introduce

$$V(S,t) = e^{-D(T-t)}V_1(S,t)$$

so that we now have

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r-D)S\frac{\partial V_1}{\partial S} - (r-D)V_1 = 0$$

which is the Black-Scholes equation only with r replaced by $r - D$ and with the same final conditions. As such

$$C(S,t) = e^{-D(T-t)}SN(d_{10}) - Xe^{-r(T-t)}N(d_{20})$$

where

$$\begin{aligned} d_{10} &= \frac{\log(S/X) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_{20} &= \frac{\log(S/X) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned}$$

6.3 Discrete dividend payments

When considering options where the underlying is a stock then a more realistic model is to treat dividends as being paid at discrete points in time. This is because most companies pay out their dividends periodically, every quarter, every six months, every year etc.

Assume, as a starting point, that just one dividend payment is made during the lifetime of the option. Assume that this is paid at time t_d and can be expressed as a percentage of the level of the underlying, i.e. as $d_y S$ where $0 \leq d_y < 1$. Thus the holder of the asset receives a payment of $d_y S$ at t_d where S is the asset price **prior** to the dividend payment. How does this affect the asset price? By the usual arbitrage arguments if t_d^- is the time immediately before the dividend is paid and t_d^+ is the time immediately after we have

$$\begin{aligned} S(t_d^+) &= S(t_d^-) - d_y S(t_d^-) \\ &= (1 - d_y) S(t_d^-) \end{aligned}$$

where $S(t)$ is the value of the underlying asset at time t. There is a **jump** in the value of S, in that the value of the underlying asset is discontinuous across the dividend date. What effect will this have on the option price? Again in order to eliminate any possible arbitrage opportunities, the value of the option must be **continuous** as a function of time across the dividend date. In which case the value of the option immediately before the dividend payment must be the same as the value immediately after (recall that the owner of the option does *not* receive the dividend) thus

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+).$$

This brings to light something interesting in the relationship between S and t. In the Black-Scholes methodology S and t are considered to be independent variables although S is clearly dependent on t, this is possible as we consider every possible value of S at a particular point in time, rather than just one. This is because given the random movement of stock prices, S can take any value.

As S is not fixed across the dividend date, in fact we know that $S(t_d^+) = (1-d_y)S(t_d^-)$ then there is no contradiction in the above relationship between $V(t_d^-)$ and $V(t_d^+)$, as we have

$$V(S, t_d^-) = V(S(1 - d_y), t_d^+).$$

So the option value is continuous across the dividend date even if the value of the underlying is discontinuous and the relationship is given above.

6.3.1 Example: pricing a European call option when there is one dividend payment

As usual we work back from the known conditions at expiry to derive the option value at a previous time. Moving backwards from expiry to just after the dividend payment time, namely t_d^+. At the dividend payment date we implement the jump condition

$$C(S, t_d^-) = C(S(1 - d_y), t_d^+).$$

then value the option back to any desired time t using these option values as new final conditions. Essentially you have to solve the Black Scholes equation twice

- Once for $T > t > t_d$ with $C(S, T) = \max(S - X, 0)$.
- Once for $t_d > t > 0$ with $C(S, t_d) = C(S(1 - d_y), t_d^+)$

We can simplify the methodology slightly by the following procedure:

Let $C(S,t)$ be the standard European call option and $C_d(S,t)$ be an option on an underlying asset paying discrete payments. If there is just one payment at t_d then from above we have

$$\begin{aligned} C_d(S,t) &= C(S,t;X), \quad t_d^+ \leq t < T \\ C_d(S,t_d^-) &= C_d(S(1-d_y),t_d^+) \\ &= C(S(1-d_y),t_d^+;X). \end{aligned}$$

For $t < t_d^-$ there is a shortcut to using the BSE. Prior to the dividend payment the value of the call option is just subject to a scaling in S, i.e $S \mapsto S(1-d_y)$ as such $C(S(1-d_y),t;X)$ still satisfies the Black-Scholes equation. As this is equal to the value of $C_d(S,t)$ at t_d then the two are also equivalent for $t < t_d$. Thus if we can find the value of $C(S(1-d_y),t;X)$ then we'll know the value of C_d for $t < t_d$ and hence for all t.

At expiry,

$$\begin{aligned} C(S(1-d_y),T;X) &= \max(S(1-d_y)-X,0) \\ &= (1-d_y)\max(S-\frac{X}{1-d_y},0) \end{aligned}$$

which is the same as $(1-d_y)$ calls with an exercise price of $X/(1-d_y)$, hence we now know the value of the call option for $0 \leq t < t_d$, which is

$$C_d(S,t) = (1-d_y)C(S,t;\frac{X}{1-d_y}).$$

In conclusion

$$C_d(S,t) = \begin{cases} (1-d_y)C(S,t;\frac{X}{1-d_y}) & \text{for} \quad 0 \leq t < t_d \\ C(S,t;X) \quad \text{for} \quad t_d \leq t < T \end{cases}.$$

which can be valued using the standard option pricing formulae.

Remark: Note that if the underlying asset pays a dividend then this *decreases* the value of the call option, since the holder of the the option does not receive the dividend yet a dividend payment reduces the value of the underlying asset. Correspondingly the value of a put option *increases* when dividends are paid.

7 American Options

American options are options which can be exercised at any time to receive $S-X$ or $X-S$ for call and put options respectively. Unfortunately this gives rise to a **non-linear** problem and as such it is not possible in general to derive explicit formulae like those for European options.

7.1 American put options

The first problem is to decide at which values of S and t it is optimal to exercise. To consider the problem, treat the American put option as a European put option with the extra early exercise feature. At expiry the early exercise condition has no effect, as the value of the American put, $P(S,t)$, is given by

$$P(S,T) = \max(X - S, 0).$$

Moving back from expiry there will, however, be certain values of S for which

$$X - S > P_{BS}(S,t)$$

where $P_{BS}(S,t)$ is the value of the European put option derived from the Black-Scholes PDE. In this case the holder of the option would exercise their right and receive $X - S$. The major problem is to locate the value of S at which it becomes optimal to exercise the option, if we call this value $S_f(t)$ then we have

$$P(S,t) = \begin{cases} X - S & \text{for} \quad S \leq S_f(t) \\ P_{BS}(S,t) & \text{for} \quad S > S_f(t). \end{cases}$$

This is known as a free boundary problem and they are very difficult to solve. More formally when pricing American options the Black-Scholes equation becomes an inequality, which is an equality when it is optimal to hold the option:

$$S_f(t) < S < \infty : \qquad P > X - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0,$$

and an inequality when it is optimal to exercise

$$0 \leq S < S_f(t) : \qquad P = X - S, \quad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0.$$

The boundary conditions are as follows:

$$\begin{aligned} P(S,T) &= \max(X - S, 0), \\ P(S_f(t), t) &= X - S_f(t), \\ P(S,t) \to 0 \quad &\text{as} \quad S \to \infty. \end{aligned}$$

where the first and third are as for a European put option but the second is one of the conditions on the free boundary, $S_f(t)$. There is another, less obvious condition at $S = S_f(t)$, known as the smooth pasting condition which ensures that the Δ $(= \partial P/\partial S)$ is smooth across the early exercise boundary, namely

$$\frac{\partial P}{\partial S}(S_f(t), t) = -1.$$

If this were not the case then there are arbitrage possibilities (see the explanation in Wilmott, Howison and Dewynne, 1995, p. 110-111)

In general, numerical methods must be used to price American put options. There is one exception though and that is the perpetual case.

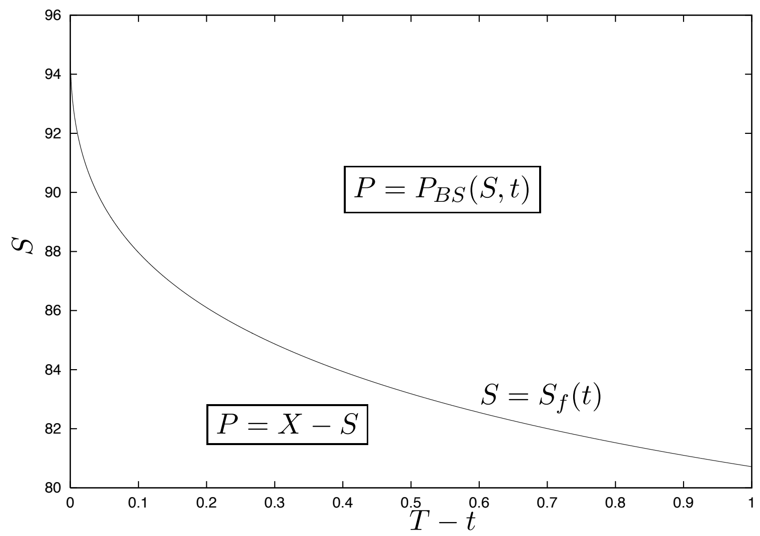

Figure 8: The position of $S_f(t)$ and the valuation regions for an American put option.

7.2 American call options

If the underlying asset pays no dividends then pricing an American call option is remarkably simple. Recall that these options can be described as a European call with the added feature that it is possible to exercise at any time to receive $S - X$. However, consider a portfolio

$$\Pi = S - C$$

where C is a *European* call option, so at expiry

$$\Pi = S - \max(S - X, 0) \leq X$$

hence, for $t < T$

$$S - C \leq Xe^{-r(T-t)}$$

or

$$C \geq S - Xe^{-r(T-t)} \geq S - X$$

thus it is **never** optimal to early exercise this American call option and so the price is the same as for a European call option. This is not the case when the underlying asset is paying continuous dividends as one can observe from the option profiles in figures 9 and 10. In the continuous dividend case the problem becomes similar to that for the American put, with analogous boundary conditions.

$$0 < S < S_f(t): \qquad C > S - X, \quad \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC = 0,$$

and with the BSE being an inequality when it is optimal to exercise

$$S_f(t) < S < \infty: \qquad C = S - X, \quad \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S\frac{\partial C}{\partial S} - rC < 0.$$

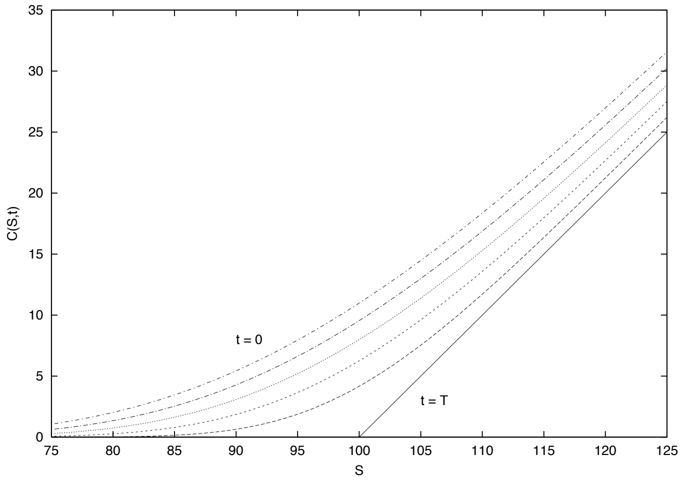

Figure 9: The value of $C(S,t)$ at $t = 0, \ldots$ and $t = T$ on a non-dividend paying asset - note how the value of $C(S,t)$ does not drop below $S - X$.

The boundary conditions are as follows:

$$\begin{aligned} C(S,T) &= \max(S - X, 0), \\ C(S_f(t), t) &= S_f(t) - X, \\ C(0,t) &= 0. \end{aligned}$$

and there is also an equivalent smooth pasting condition:

$$\frac{\partial C}{\partial S}(S_f(t), t) = 1.$$

7.3 Perpetual options

These are options with an infinite life, corresponding to $T \to \infty$. In this case we look for solutions (for American puts) of the form $P(S)$ only. The Black-Scholes equation then becomes the following ODE:

$$\tfrac{1}{2}\sigma^2 S^2 \frac{d^2P}{dS^2} + rS\frac{dP}{dS} - rP = 0.$$

This is a form of Euler's equation, and hence has solutions of the form $P = AS^{\alpha}$, where

$$\tfrac{1}{2}\sigma^2 \alpha(\alpha - 1) + r\alpha - r = 0,$$

and solving this (quadratic) equation for α yields two values, $\alpha = 1$ or $\alpha = -\frac{2r}{\sigma^2}$.

The conditions to be satisfied are that

$$P(S \to \infty) \to 0, \quad P(S = S_f) = X - S_f, \quad \frac{dP}{dS}(S = S_f) = -1.$$

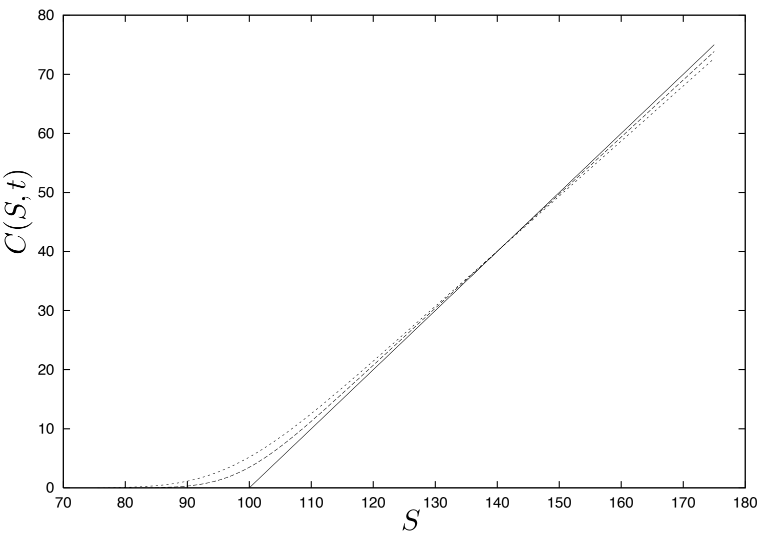

Figure 10: The value of $C(S,t)$ at $t=0, \ldots$ and $t=T$ on a dividend paying asset - note how the value of $C(S,t)$ can drop below $S-X$.

The first of these conditions indicates we can discard the $\alpha = 1$ solution, and so

$$P = AS^{-2r/\sigma^2}.$$

The smooth pasting conditions lead to

$$X - S_f = AS_f^{-2r/\sigma^2},$$

$$-1 = -\frac{2r}{\sigma^2} AS_f^{-2r/\sigma^2 - 1},$$

which lead to the location of the free boundary

$$S_f = \frac{X}{\frac{\sigma^2}{2r} + 1}.$$

Again it is possible to value a perpetual call options with dividends by using simple ODE theory together with the relevant boundary conditions.

8 Interest rate models and bonds

So far we have assumed that interest rates are constant or at best known functions of time; this is clearly not the case in reality. Although the effects of interest-rate changes on option prices are generally small (because of their short lifetime), many other securities with much longer durations can be very susceptible to interest rate changes.

8.1 Bonds

A bond is a contract, paid for up-front, that yields a known amount on a known date in the future, the **maturity date**, $t = T$. The bond may also pay a known cash dividend (the **coupon**) at fixed times during the life of the contract. If there are no coupons, the bond is known as a **zero-coupon bond**. Bonds may be issued by both governments and companies to raise capital, and the up-front premium can be regarded as a loan.

A typical question related to this is: *how much should I pay now to get a guaranteed \$1 in 10 years' time?*

In the simple case of a zero-coupon bond $V(t)$ which pays Z at $t = T$ we may equate the return to that of a bank deposit, i.e.

$$dV = r(t)V dt,$$

with $V(T) = Z$. If the interest rate is deterministic, then

$$V(r, t; T) = Ze^{-\int_t^T r(\tau)d\tau}.$$

IF the bond pays a single coupon ('dividend') amount Z_1 at $t = T_1 < T$, then the net effect is that of an additional 'mini' bond maturing at $t = T_1$, in addition to the main bond. The value overall for $t < T_1$ is then modified as follows:

$$V(r, t < T_1; T_1; T) = Ze^{-\int_t^T r(\tau)d\tau} + Z_1e^{-\int_t^{T_1} r(\tau)d\tau},$$

whilst for $t > T_1$ the value is unaffected, i.e.

$$V(r, t > T_1; T_1; T) = Ze^{-\int_t^T r(\tau)d\tau}.$$

8.2 Stochastic interest rates

In the same way we developed a model for the asset price as a lognormal walk, suppose that the interest rate r is governed by a stochastic differential equation

$$dr = w(r, t)dX + u(r, t)dt.$$

The functional form of $w(r, t)$ and $u(r, t)$ determines the behaviour of the **spot rate** r.

8.3 The bond-pricing equation

Pricing a bond is trickier than pricing an option, since there is no underlying asset with which to hedge: we cannot go out and buy an interest rate of 5%. Instead, we hedge with bonds of different maturity dates.

We set up a portfolio comprising two bonds with different maturities T_1 and T_2, namely V_1 and V_2 respectively. We hold one V_1 bond and $-\Delta$ of V_2 bonds, and so

$$\Pi = V_1 - \Delta V_2.$$

Using the above stochastic differential equation for the interest rate, in conjunction with Ito's Lemma, gives the change in this portfolio in a time dt:

$$\begin{aligned} d\Pi = &\frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt \\ &-\Delta(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt). \end{aligned}$$

From this we see that the choice

$$\Delta = \frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r}$$

eliminates the random component of $d\Pi$. We then have

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r}(\frac{\partial V_2}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2})\right)dt \\ &= r\Pi dt \\ &= r(V_1 - V_2\frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r})dt, \end{aligned}$$

where we have used arbitrage arguments to set the return on the portfolio to equal the risk-free (spot) rate.

Gathering all the V_1 terms on the left-hand-side and all the V_2 terms on the right-hand-side yields

$$(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1)/\frac{\partial V_1}{\partial r} = (\frac{\partial V_2}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2)/\frac{\partial V_2}{\partial r}$$

This is one equation in two unknowns, however the left-hand-side is a function of T_1 but not T_2, and the right-hand-side is a function of T_2 but not T_1. The only way that this is possible is for both sides to be independent of the maturity date. Thus

$$(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV)/\frac{\partial V}{\partial r} = a(r,t)$$

for some function $a(r,t)$. It is convenient to write

$$a(r,t) = w(r,t)\lambda(r,t) - u(r,t)$$

for given $w(r,t)$ and $u(r,t)$, but $\lambda(r,t)$ unspecified.

The zero-coupon bond pricing equation is therefore

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0,$$

subject to the final condition $V(r,T) = Z$, and generally $V(r \to \infty, t) \to 0$; the boundary condition on $r = 0$ is generally dependent on λ, u and w.

8.4 The market price of risk

Consider now in more detail the unknown function $\lambda(r,t)$. In a timestep dt the bond V changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + (\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r})dt.$$

From the PDE derived above for V we can rewrite the bracketed term, giving

$$dV = w\frac{\partial V}{\partial r}dX + (w\lambda\frac{\partial V}{\partial r} + rV)dt,$$

or

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt).$$

The presence of dX indicates this is not a risk-less portfolio. The right-hand-side is the excess return above the risk-free rate for accepting a certain level of risk. In return for taking the extra risk the portfolio profits by an extra λdt per unit of extra risk dX. The function λ is called the **market price of risk**.

8.5 The Vasicek model

This takes the form

$$dr = (\eta - \gamma r)dt + \beta^{\frac{1}{2}}dX$$

The model is tractable - explicit formulae exist. For a zero-coupon bond, value is

$$e^{A(t;T)-rB(t;T)}$$

Substituting into the PDE, and considering the $O(r^0)$ and $O(r)$ terms separately (see examples 8), yields

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

$$A = \frac{1}{\gamma^2}(B - T + t)(\eta\gamma - \lambda\gamma\beta^{\frac{1}{2}} - \tfrac{1}{2}\beta) - \frac{\beta B^2}{4\gamma}$$

Model is mean reverting (which is good), but interest rates can go negative (which is bad).

8.6 Cox, Ingersoll, Ross Model

The CIR model takes the form

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX$$

Spot rate is mean reverting, and remains positive if $\eta > \alpha/2$. For a zero-coupon bond, value is again of the form

$$A(t;T)e^{-rB(t;T)}$$

Substituting into the PDE, and considering the $O(r^0)$ and $O(r)$ terms separately, yields

$$\frac{dA}{dt} = \eta A(r)B(t)$$

$$\frac{dB}{dt} = (\gamma + \lambda\sqrt{\alpha}B + \tfrac{1}{2}\alpha B^2 - 1$$

with $A(T) = 1,\ B(T) = 0$

The solution is given by

$$A(t) = \left\{ \frac{2\xi e^{(\xi+\psi)(T-t)/2}}{(\xi+\psi)(e^{\xi(T-t)} - 1) + 2\xi} \right\}^{2\eta/\alpha}$$

$$B(t) = \frac{2(e^{\xi(T-t)} - 1)}{(\xi+\psi)(e^{\xi(T-t)} - 1) + 2\xi}$$

where $\psi = \gamma + \lambda\sqrt{\alpha}$, $\xi = \sqrt{\psi^2 + 2\alpha}$

9 Barrier options

Barrier options are path dependent options - they have a payoff that depends on the realised asset price via its level; certain aspects of the contract are triggered if the asset price becomes too high or too low.

Example: An up-and-out call option pays off the usual $\max(S - X, 0)$ at expiry unless at any time previously the underlying asset has traded at a value S_u or higher. If the asset reaches this level (obviously from below) then it is 'knocked out', becoming worthless. As well as 'out' options, there are also 'in' options which only receive a payoff if a level is reached, otherwise they expire worthless.

Barrier options are useful for a number of reasons, including

(i) The purchaser has precise views about the direction of the market.

(ii) The purchaser wants the payoff from an option, but does not want to pay for the upside potential, believing that the movement of the underlying will be limited prior to expiry.

(iii) These options are cheaper than their corresponding vanilla 'cousins'.

9.1 Pricing barrier options with PDEs

Although barrier options are path dependent, this dependency can be quite readily incorporated into the PDE methodology - we only need to know whether or not the barrier has been triggered; we do not need any other information about the path. This is in contrast to other more exotic types of option, such as Asian options (where, for example, the payoff may depend on the average value of the underlying during the lifetime of the option contact).

Consider the value of a barrier contract before the barrier has been triggered. The value still satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

9.2 Out barriers

If the underlying reaches the barrier in an 'out' barrier option, then the contract becomes worthless. This leads to the boundary condition

$$V(S_u, t) = 0 \quad \text{for} \quad t < T,$$

for an up-and-out barrier option with the barrier level at $S = S_u$. We must solve the Black-Scholes equation for $0 \le S \le S_u$ with the above condition on $S = S_u$ and the usual payoff condition if the barrier is not triggered.

If we have a down-and-out option with a barrier at S_d, then we solve for $S_d \le S < \infty$ with

$$V(S_d, t) = 0,$$

and the relevant final condition at expiry.

9.3 In barriers

An 'in' barrier option only has a payoff if the barrier is triggered. If the barrier is not triggered, then the option expires worthless. The value in the option is the potential to hit the barrier. If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a vanilla contract (say $V_v(S,t)$). We then have

$$V(S_u, t) = V_v(S_u, t) \quad \text{for} \quad t < T.$$

A similar boundary condition holds for a down-and-in option.

The contract we receive when the barrier is triggered is a derivative itself, and therefore the 'in' option is a second-order contract. We must therefore solve for the vanilla option first, before solving for the value of the barrier option.

9.4 Down-and-out call options

Consider the down-and-out call option with barrier level S_d below the strike price X. The function $V_v(S,t)$ is the Black-Scholes value of the corresponding vanilla option.

It is easy (see examples 8) to show that

$$V = S^{1-2r/\sigma^2} V_v(A/S, t)$$

also satisfies the Black-Scholes equation for any A (constant).

From this we can infer the value of a down-and-out call option, namely

$$V(S,t) = V_v(S,t) - (\frac{S}{S_d})^{1-2r/\sigma^2} V_v(S_d^2/S, t).$$

We can confirm this is the solution (from the above we know this will satisfy the Black-Scholes equation). If we substitute $S = S_d$, we find $V(S_d, t) = 0$. Since $S_d^2/S < X$ for $S > S_d$, the value of $V_v(S_d^2/S, T)$ is zero; thus the final condition is satisfied.

9.5 Down-and-in call options

The relationship between an 'in' barrier option and an 'out' barrier option (with the same payoff and barrier level) is very simple

$$\text{in} + \text{out} = \text{vanilla}.$$

If the 'in' barrier is triggered, then so is the 'out' barrier, so whether or not the barrier is triggered, we still obtain the vanilla payoff at expiry. Thus the value of a down-and-in call option is

$$(\frac{S}{S_d})^{1-2r/\sigma^2} V_v(S_d^2/S, t).$$