5 Solving the heat conduction and Black-Scholes equations

The PDE which defines the price of a derivative is now known to be a second-order **parabolic** equation, in the majority of cases this equation is also a linear one. This chapter is concerned with the nature of these equations, focusing attention on the heat conduction equation and then extending to the Black-Scholes equation itself.

5.1 Properties of the Heat conduction equation

The heat conduction equation takes the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where τ is the time and x is the spatial variable, it normally models the flow of heat or its diffusion and has been extensively studied over the years. Its fundamental properties are as follows

- It is a second order linear PDE, as such if u_1 and u_2 are solutions then so is $a_1u_1 + a_2u_2$ for any constants a_1, a_2
- It is a parabolic equation and it's characteristics are simply along the lines $\tau = c$ (where c is a constant) which means that this is where information propagates along. So any change in the boundary conditions is felt along these lines.
- The heat conduction equation generally has analytic solutions in x, technically in that for $\tau > 0$, $u(x, \tau)$ has a convergent power series of $(x - x_0)$ for $x_0 \neq x$.

Crucially, the heat conduction (diffusion) equation is a smoothing out process, and as such discontinuities in the boundary or initial (final) conditions can be catered for. Recall that in the Black-Scholes equation the final conditions are often discontinuous.

Example By way of demonstration consider the following *initial value problem.*

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

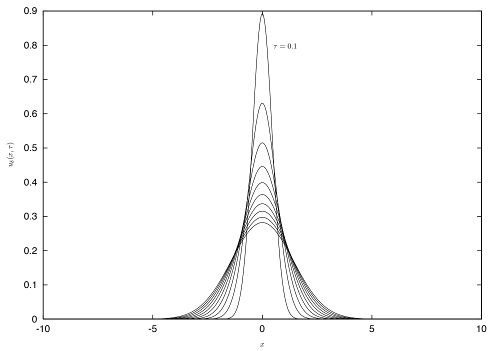

Figure 5: A graphical representation of $u_\delta(x,\tau)$ for $\tau = 0.1, 0.2, 0.3, \ldots, 1$.

for $\tau > 0$ and $-\infty < x < \infty$ where $u(x,0) = u_0(x)$ and $u \to 0$ as $x \to \pm\infty$. $u(x,\tau)$ is analytic for $\tau > 0$. Consider a special solution, about which more is said later

$$u(x,\tau) = u_\delta(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/4\tau} \tag{29}$$

for $-\infty < x < \infty$ and $\tau > 0$. Now we verify that this indeed satisfies the PDE.

$$\frac{\partial u}{\partial x} = \frac{-x}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-1}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau} + \frac{x^2}{8\tau^{5/2}\sqrt{\pi}} e^{-x^2/4\tau}$$

$$\frac{\partial u}{\partial \tau} = \frac{-1}{4\tau^{3/2}\sqrt{\pi}} e^{-x^2/4\tau} + \frac{x^2}{8\tau^{5/2}\sqrt{\pi}} e^{-x^2/4\tau}.$$

So, this is a solution which is well behaved except at one instance, the initial point in time $\tau = 0$. At this point when $x \neq 0$ then $u_\delta(x,0) = 0$ but at $x = 0$ it has infinite value. This clearly has discontinuous initial conditions yet gives rise to a, reasonably, well behaved solution.

What more can we say about this special solution to the heat conduction equation? Well,

$$\int_{-\infty}^{\infty} u_\delta(x,\tau)dx = 1, \quad \forall\tau.$$

This function has all of the *heat* initially ($\tau = 0$) concentrated at $x = 0$ and then this immediately dissipates out as for *any* $\tau > 0$, $u_\delta(x,\tau) > 0$ for *all* values of x.

Finally note the close similarity between the probability density function for the Normal distribution ($\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$) and the value of $u_\delta(x,\tau)$. Clearly it is the same only with a mean(μ) of zero and a variance (σ^2) of 2τ. As such it is possible to interpret this particular solution as the probability density function of the future position of a particle following a Brownian motion ($\sqrt{2}dW$) along the x-axis, with the particle starting at the origin.

5.2 The Dirac delta function

The function $u_\delta(x, \tau)$ when $\tau = 0$ is one representation of the **(Dirac) delta function** which is not a function in the normal sense but is known as a **generalised** function. It's definition is as a linear map representing the limit of a function whose effect is confined to a smaller and smaller interval but remains finite.

An informal definition is to consider a function

$$f(x) = \begin{cases} 1/2\epsilon, & |x| \leq \epsilon \\ 0, & |x| > \epsilon \end{cases}$$

and as $\epsilon \to 0$ the graph becomes taller and narrower but at all points

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

regardless of the value of ϵ although for all $x \neq 0$, $f(x) \to 0$ as $\epsilon \to 0$. In general the delta function $\delta(x)$ is the limit as $\epsilon \to 0$ of any one-parameter family of functions δ_ϵ with the following properties

- for each ϵ, $\delta_\epsilon(x)$ is piecewise smooth;
- $\int_{-\infty}^{\infty} \delta_\epsilon(x)dx = 1$;
- for each $x \neq 0$, $\lim_{\epsilon \to 0} \delta_\epsilon(x) = 0$.

Note that the specific solution to the heat conduction equation u_δ satisfies the above constraints with τ replaced by ϵ. The best way to look at the delta function is to only consider its integral which we know to be 1 and which smooths out the function's bad behaviour, especially when $x = 0$ and $\epsilon \to 0$ (of $\tau \to 0$). When concentrating on the integral form we can see the delta function as a **test function**, in that

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x)\phi(x)dx &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \delta_\epsilon(x)\phi(x)dx \\
&= \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{-\epsilon} \delta_\epsilon(x)\phi(x)dx + \int_{-\epsilon}^{\epsilon} \delta_\epsilon(x)\phi(x)dx + \int_{\epsilon}^{\infty} \delta_\epsilon(x)\phi(x)dx \right\} \\
&= \lim_{\epsilon \to 0} \left\{ \phi(0) \int_{-\epsilon}^{\epsilon} \delta_\epsilon(x)dx \right\} \\
&= \phi(0)
\end{aligned}$$

In fact, for any $a, b > 0$

$$\int_{-a}^{b} \delta(x)\phi(x)dx = \phi(0)$$

and, as importantly, for any x_0

$$\int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0)$$

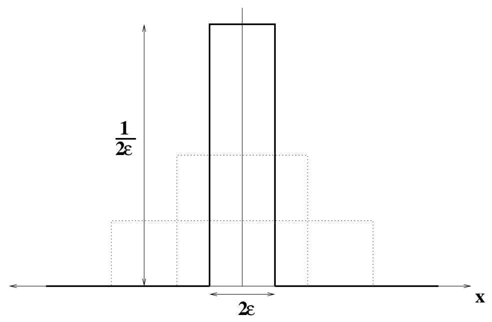

Figure 6: The epsilon representation of $\delta(x)$ which is the limit as $\epsilon \to 0$.

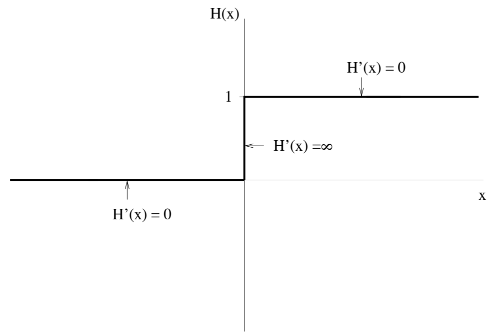

Figure 7: Demonstration that $H'(x) = \delta(x)$.

and so integrating picks out the value of ϕ at x_0, the reason why $\delta(x)$ is also known as a test function.

Other properties concern its links with the Heaviside function as

$$\int_{-\infty}^{x} \delta(s)ds = \mathcal{H}(x)$$

and conversely,

$$\mathcal{H}'(x) = \delta(x)$$

where, as before

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

5.3 Transforming the Black-Scholes equation

Consider the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

make the following three substitutions

$$\begin{aligned} S &= Xe^x (\text{or} \quad x = \log\frac{S}{X}) \\ t &= T - \frac{\tau}{\frac{1}{2}\sigma^2} (\text{or} \quad \tau = \frac{\sigma^2}{2}(T-t)) \\ V &= Xv(x,\tau) \end{aligned} \tag{30}$$

thus

$$\begin{aligned} \frac{\partial V}{\partial t} &= X\frac{\partial v}{\partial \tau}\frac{d\tau}{dt} = X\frac{\partial v}{\partial \tau}. - \frac{\sigma^2}{2} = -\frac{X\sigma^2}{2}\frac{\partial v}{\partial \tau} \\ \frac{\partial V}{\partial S} &= X\frac{\partial v}{\partial x}\frac{dx}{dS} = X\frac{\partial v}{\partial x}\frac{1}{S} = e^{-x}\frac{\partial v}{\partial x} \\ \frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{e^{-x}}{X}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-x}}{X}\left(e^{-x}\frac{\partial^2 v}{\partial x^2} - e^{-x}\frac{\partial v}{\partial x}\right) = \frac{e^{-2x}}{X}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right) \end{aligned}$$

which leads to

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$$

where

$$k = \frac{r}{\frac{1}{2}\sigma^2}$$

Now attempt to remove the $\frac{\partial v}{\partial x}$ and v terms by introducing the substitution

$$v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$$

where α and β are constants to be determined, this gives

$$\begin{aligned} \frac{\partial v}{\partial \tau} &= \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} \\ \frac{\partial v}{\partial x} &= \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} \\ \frac{\partial^2 v}{\partial x^2} &= \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

which gives

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku$$

to remove the $\frac{\partial u}{\partial x}$ and u terms we require

$$\begin{aligned} \alpha &= -\frac{1}{2}(k-1) \\ \beta &= -\frac{1}{4}(k+1)^2. \end{aligned}$$

Thus,

$$V(S,t) = Xe^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \tag{31}$$

and

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

To transform the final conditions, or the payoff from the option we have for a **call** option

$$V(S,T) = \max(S-X,0)$$

so, from the definition of x, τ and $v(x,\tau)$ in (30)

$$Xv(x,0) = \max(Xe^x - X, 0)$$

or

$$v(x,0) = \max(e^x - 1, 0)$$

and so, from (31)

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right] \tag{32}$$

and similarly for a put option

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0\right] \tag{33}$$

As such the Black-Scholes equation has been converted to the heat conduction equation for $-\infty < x < \infty$ and, for European call and put options, initial condition $u_0(x)$ from (32) and (33) above. If we can determine a procedure for valuing the initial value problem for the heat conduction equation we'll be able to determine the correct values for call and put options.

5.4 Similarity solutions to the Heat conduction equation

Explanation is first by way of two examples

Example 5.1: Suppose that $u(x,\tau)$ satisfies the heat conduction equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x, \tau > 0$$

with the following boundary conditions

$$\begin{aligned} u(x,\tau=0) &= 0 && (34)\\ u(x=0,\tau) &= 1 && (35)\\ u(x,\tau)\to 0 \quad &\text{as} \quad x\to\infty && (36) \end{aligned}$$

i.e. the bar initially has heat zero and then immediately the heat at one end is raised to 1 and kept there.

Seek a solution of the form $u(x,\tau) = U(\xi)$ where $\xi = x/\sqrt{\tau}$ on substitution

$$\frac{\partial u}{\partial \tau} = \frac{dU}{d\xi}\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}x\tau^{-3/2}\frac{dU}{d\xi}$$

$$\frac{\partial u}{\partial x} = \frac{dU}{d\xi}\frac{\partial \xi}{\partial x} = \tau^{-1/2}\frac{dU}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \tau^{-1/2}\frac{d}{d\xi}\left(\tau^{-1/2}\frac{dU}{d\xi}\right) = \tau^{-1}\frac{d^2U}{d\xi^2}$$

and so, replacing $x/\sqrt{\tau}$ by ξ and multiplying by τ gives the ODE

$$\frac{d^2U}{d\xi^2} + \frac{1}{2}\xi\frac{dU}{d\xi} = 0$$

the boundary conditions become

$$U(0) = 1$$

and

$$U(\infty) = 0$$

with this second condition catering for both the initial condition and $u(x,\tau)\to 0$ as $x\to\infty$. Integrating the ODE once gives

$$\frac{dU}{d\xi} = Ce^{-\xi^2/4}$$

(C constant) and on solving gives

$$U(\xi) = C\int_0^{\xi} e^{-s^2/4}ds + D$$

(D constant). Upon substituting the boundary conditions, first $U(0) = 1$ gives

$$1 = D$$

and then $U(\infty) = 0$ gives

$$0 = C \int_0^\infty e^{-s^2/4} ds + 1$$

but we know that

$$\int_0^\infty e^{-s^2/4} ds = \sqrt{\pi}$$

thus

$$-1 = C\sqrt{\pi}$$

Thus,

$$U(\xi) = -\frac{1}{\sqrt{\pi}} \int_0^\xi e^{-s^2/4} ds + 1$$

but

$$\int_0^\xi = \int_0^\infty - \int_\xi^\infty$$

hence

$$U(\xi) = -\frac{1}{\sqrt{\pi}} \left(\int_0^\infty e^{-s^2/4} ds - \int_\xi^\infty e^{-s^2/4} ds \right) + 1$$

or

$$U(\xi) = -\frac{1}{\sqrt{\pi}} \left(- \int_\xi^\infty e^{-s^2/4} ds \right) - 1 + 1$$

so

$$U(\xi) = \frac{1}{\sqrt{\pi}} \int_\xi^\infty e^{-s^2/4} ds$$

and on replacing ξ by its definition we get

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{\tau}}^\infty e^{-s^2/4} ds$$

The key trick being that to solve the equation we replace two variables (x and τ) by just one (ξ) and then the problem reduces to an ODE. Even more useful is the next example, for $-\infty < x < \infty$.

Example 5.2: Consider the following equation for $u(x,\tau)$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

where

$$\int_{-\infty}^\infty u(x,\tau) dx = k, \forall \tau \quad \text{where } k \text{ is a constant.}$$

Choosing the normalised case where $k = 1$ we search for a solution of the form $u(x,\tau) = \tau^{-1/2} U(\xi)$ where $\xi = x/\sqrt{\tau}$. The other boundary condition is a somewhat odd one but is that as $|\xi| \to \infty$ then

$$U(\xi) = o(1/\xi)$$

which says that the solution must decay faster than $1/\xi$ as ξ gets very big (or alternatively $u(x,\tau) = o(1/x)$ as $|x| \to \infty$). On transforming the derivatives we get

$$\frac{\partial u}{\partial \tau} = -\frac{1}{2}\tau^{-3/2}U + \tau^{-1/2}\frac{dU}{d\xi}. - \frac{1}{2}x\tau^{-3/2} = -\frac{1}{2}\tau^{-3/2}U - \frac{1}{2}\xi\tau^{-3/2}\frac{dU}{d\xi}$$

$$\frac{\partial u}{\partial x} = \tau^{-1/2}\frac{dU}{d\xi}\frac{\partial \xi}{\partial x} = \tau^{-1}\frac{dU}{d\xi}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \tau^{-1/2}\frac{d}{d\xi}\left(\tau^{-1}\frac{dU}{d\xi}\right) = \tau^{-3/2}\frac{d^2U}{d\xi^2}$$

which gives

$$\frac{d^2U}{d\xi^2} + \frac{1}{2}\xi\frac{dU}{d\xi} + \frac{1}{2}U = 0$$

or

$$\frac{d^2U}{d\xi^2} + \frac{d}{d\xi}\left(\frac{1}{2}\xi U\right) = 0.$$

Integrating both sides wrt ξ gives

$$\frac{dU}{d\xi} + \frac{1}{2}\xi U = C$$

where C is a constant. Now as $\xi \to \infty$, $U = o(1/\xi)$ so the LHS is $o(1)$ thus this constant $C = 0$. So then on solving the ODE

$$U(\xi) = Ae^{-\xi^2/4},$$

where A is a constant. Putting in the condition we have

$$A\int_{-\infty}^{\infty} \tau^{-1/2}e^{-x^2/4\tau}dx = 1$$

however, set $x' = x/\sqrt{\tau}$ and we get $dx = \sqrt{\tau}dx'$ and the equation becomes

$$A\int_{-\infty}^{\infty} e^{-x'^2/4}dx' = 1$$

and so using the usual result

$$2A\sqrt{\pi} = 1$$

thus

$$A = \frac{1}{2\sqrt{\pi}}$$

and so,

$$u(x,\tau) = \tau^{-1/2}\left(\frac{1}{2\sqrt{\pi}}e^{-x^2/4\tau}\right)$$

or

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}e^{-x^2/4\tau}$$

which is precisely the special solution u_δ from section 5.1, equation 29. [**Note:** The derivation in Wilmott where he states that $U(\xi) = Ce^{-\xi^2/4} + D$ is *wrong*.]

5.4.1 How similarity solutions work

The reason why the above similarity solution worked was because the governing equations and the boundary conditions do not change under the scalings $x \to \lambda x$ and $\tau \to \lambda^2 \tau$, where $\lambda \in \mathbf{R}$. In particular consider new variables $x^* = \lambda x$ and $\tau^* = \lambda^2 \tau$, these clearly satisfy the heat-conduction equation and in Example 5.1 the boundary conditions become $u(x^*, 0) = 0$ and $u(0, \tau^*) = 1$ for *any* λ.

Combining these two results to get a variable which is independent of λ the only possible combination is $x/\sqrt{\tau} = x^*/\sqrt{\tau^*}$. Hence the solution to the problem must be a function of $x/\sqrt{\tau}$ only.

Similarity solutions only work in special cases where all the boundary and initial conditions are invariant under the scaling transformation. It is also possible to multiply $U(\xi)$ by a function of τ as in Example 5.2 because as the heat-conduction equation is linear it is invariant under the scaling $u \to \mu u$.

In general with similarity solutions a good practical test to see if they'll work is to search for a solution of the form $u = \tau^\alpha U(x\tau^\beta)$ in the hope that the PDE will reduce to an ODE in $\xi = x\tau^\beta$ **and** the boundary conditions will be satisfied. For the heat conduction equation then in all cases $\beta = -1/2$ but the value of α will be dependent on the specific boundary conditions. For example in 5.1 $\alpha = 0$ because of the condition at $x = 0$ and, in Example 5.2, $\alpha = -1/2$ to remove τ from the integral condition.

5.5 General solution to the Heat-Conduction equation initial value problem

Searching for a solution to the initial value problem in which we have to solve

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

with initial data $u(x, 0) = u_0(x)$ and there are suitable growth conditions at $|x| \to \infty$ (usually $\lim_{|x|\to\infty} u(x, \tau)e^{-ax^2} = 0$ for $a > 0$ and $\tau > 0$).

The key to the formulation is the delta function, $\delta(x)$ as we can write the initial conditions as

$$u_0(x) = \int_{-\infty}^{\infty} u_0(\xi)\delta(\xi - x)d\xi$$

we recall that the fundamental solution to the initial value problem from 5.2 is

$$u_\delta(s, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-s^2/4\tau}$$

and has initial value $u_\delta(s, 0) = \delta(s)$. Noting that because $u_\delta(s - x, \tau) = u_\delta(x - s, \tau)$ we have

$$u_\delta(s - x, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-(s-x)^2/4\tau}$$

which is still a solution to the heat conduction equation with either s or x as the spatial independent variable and it has initial value

$$u_\delta(s-x,0) = \delta(s-x).$$

Now comes the important bit, hence, for each s the function

$$u_0(s)u_\delta(s-x,\tau)$$

as a function of x and τ with s held fixed, satisfies the heat conduction equation as $u_0(s)$ is simply a constant. Now using the fact that the diffusion equation is linear we can add together linear combinations of these solutions for any s all the way from $-\infty$ to ∞ and obtain another solution to the heat conduction equation, namely

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds$$

and the initial data is

$$u(x,0) = \int_{-\infty}^{\infty} u_0(s)\delta(s-x)ds = u_0(x).$$

What does all this mean? Well, this solution satisfies the heat conduction equation for all x and for $\tau > 0$ and is also satisfies the initial conditions for *all* initial conditions $u_0(x)$. It is also possible to show that this solution is unique (see Examples 5). Hence we have found the general solution.

5.6 Pricing European call and put options

We now know the general solution to the initial value problem for the heat conduction equation, where $u(x,0) = u_0(x)$ for $\tau > 0$ and $-\infty < x < \infty$, namely

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(x-s)^2/4\tau}ds.$$

We start by valuing a European call option but the procedure is similar for a put option. In section 5.3 we transformed the European call option pricing problem to the following system

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{l} -\infty < x < \infty \\ \tau > 0 \end{array}$$

where

$$u(x,0) = u_0(x) = \max\left[e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right].$$

By using the known general solution to this problem we have

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\left\{\max[e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s}, 0]e^{-(x-s)^2/4\tau}\right\}ds$$

but $u_0(x) = 0$ for $x < 0$ hence

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty}\left\{[e^{\frac{1}{2}(k+1)s} - e^{\frac{1}{2}(k-1)s}]e^{-(x-s)^2/4\tau}\right\}ds.$$

We make another change of variable, define

$$x' = \frac{s-x}{\sqrt{2\tau}}.$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\{\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)-\frac{1}{2}x'^2}dx' - \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)-\frac{1}{2}x'^2}dx'\}.$$

Completing the square and removing the terms not dependent on x' yields

$$\begin{aligned}
u(x,\tau) &= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(x'-\frac{1}{2}(k+1)\sqrt{2\tau})^2}dx' \\
&\quad - \frac{e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(x'-\frac{1}{2}(k-1)\sqrt{2\tau})^2}dx' \\
&= I_1 - I_2 \qquad (37)
\end{aligned}$$

Noting that the expression for the cumulative Normal distribution is as follows

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}s^2}ds$$

we transform the dependent variable, x', once again to

$$x_1 = x' - \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$x_2 = x' - \frac{1}{2}(k-1)\sqrt{2\tau}$$

in I_1 and I_2 respectively and then

$$u(x,\tau) = e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1) - e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(d_2)$$

where

$$\begin{aligned}
d_1 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau} \\
d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}.
\end{aligned}$$

Transforming the variables back using the usual definitions

$$\begin{aligned}
V(S,t) &= Xe^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \\
x &= \log\left(\frac{S}{X}\right) \\
\tau &= \frac{\sigma^2}{2}(T-t) \\
k &= \frac{2r}{\sigma^2}
\end{aligned}$$

gives the following expression for the value of the European call option

$$C(S,t) = V(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2 = \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

The European put can be valued in a similar manner or, more easily, by use of the put-call parity, equation. Either approach yields the following expression for its value, $P(S, t)$

$$P(S, t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1).$$

(To use put-call parity note that $N(x) + N(-x) = 1$).