4 The Black-Scholes analysis

4.1 Converting a stochastic process to a deterministic one

In the previous section we have defined a particular model for the movement of stock prices. This is by no means the only possible process used for underlying assets but is the one which is used for the Black-Scholes analysis, which still remains the most popular model for practitioners. From here we now proceed to derive the Black-Scholes PDE.

The main problem with the process followed by the function of S, F, is that there is still a random term present which makes constructing a PDE somewhat problematic. The solution to this is to create a new function g which is completely deterministic. Consider a function

$$g = f - \Delta S$$

where Δ is an as yet unknown parameter which is constant across a time period dt. In which case the change in the value of g over this period is

$$dg = df - \Delta dS$$

and by substituting in the expressions for df and dS from equations (8) and (5) we obtain

$$\begin{aligned} dg &= \left[\mu S \tfrac{\partial f}{\partial S} + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 f}{\partial S^2}\right] dt + \sigma S \tfrac{\partial f}{\partial S} dW - \Delta[\mu S dt + \sigma S dW] \\ &= \sigma S \left[\tfrac{\partial f}{\partial S} - \Delta\right] dW + \left[\mu S \left(\tfrac{\partial f}{\partial S} - \Delta\right) + \tfrac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \tfrac{\partial^2 f}{\partial S^2}\right] dt \end{aligned}$$

Thus, if we choose

$$\Delta = \frac{\partial f}{\partial S}$$

then the equation reduces to one which has only deterministic variables. This is the basis of the technique employed by Black and Scholes to derive their PDE

4.2 The Black-Scholes PDE

Notation:

- S is the current value of the underlying asset, can also be denoted by S_t especially in SDEs but the t is usually dropped.
- t is the time elapsed since the option was created and the option expires at time T.
- $V(S, t)$ is the value of either a call or a put option.
- $C(S, t)$ is the value of a call option.
- $P(S, t)$ is the value of a put option.
- X is the exercise price of the option.

- σ is the volatility of the underlying asset or a measure of the uncertainty of its movements. For example, a telecommunications startup company's shares will have a higher volatility than Tesco's shares.
- μ is the drift of the underlying asset.
- r is the risk-free interest rate, the return that you would receive from a risk-free investment such as a government bond.

Black-Scholes assumptions:

- The underlying asset follows geometric Brownian motion ($dS = \mu S dt + \sigma S dW$) with constant drift, μ and volatility σ. It is possible to have the volatility dependent on time but more complicated models will provide much more challenging problems.
- It is permitted to short sell the underlying asset, i.e. sell an asset that you don't actually own.
- There are no transaction costs, all securities are perfectly divisible and trading takes place continuously.
- There are no dividends, or equivalent, paid out during the lifetime of the option (this will be relaxed at a later date).
- There are no riskless arbitrage opportunities. Any that do exist exist only for a very short period of time.
- The risk free rate r is constant. This can also be trivially relaxed to let r be a function of time. In practice, especially for long-term derivatives, the interest rate is itself modelled stochastically.

As the option price $V(S,t)$ depends on the underlying asset, S, which follows geometric Brownian motion

$$dS = \mu S dt + \sigma S dW \tag{9}$$

and by Itô's lemma we have

$$dV = \left[\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S \frac{\partial V}{\partial S} dW \tag{10}$$

Now construct a portfolio which consists of an option and short in Δ of the underlying. Π is defined to be the value of the portfolio where

$$\Pi = V - \Delta S. \tag{11}$$

Assume, across a time period dt, that the value of Δ is held fixed giving

$$d\Pi = dV - \Delta dS, \tag{12}$$

and so, on substituting in the expressions for dV and dS in equations (9) and (10) we get

$$d\Pi = \sigma S \left[\frac{\partial V}{\partial S} - \Delta\right] dW + \left[\mu S \left(\frac{\partial V}{\partial S} - \Delta\right) + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt \tag{13}$$

The amount of the underlying which the holder of the portfolio is short selling, Δ, has not yet been set. However, if Δ is selected, as before, such that

$$\Delta = \frac{\partial V}{\partial S}, \tag{14}$$

then the stochastic differential equation for $d\Pi$ becomes **deterministic**, as the coefficient of the dW term is now identically zero. Thus this portfolio is perfectly hedged as it provides a guaranteed return over a designated time period. Obviously, this assumes that it is possible to change the value of Δ continuously, because as time evolves the value of $\frac{\partial V}{\partial S}$ is changing. With this continuous rebalancing of the portfolio the expression for $d\Pi$ is now

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{15}$$

However, this portfolio is perfectly hedged, in that it yields a risk-less value after any period of time t and, as such, should return the risk-free rate. Assuming no arbitrage then over a period of time, dt, and a constant risk-free interest rate, r, the change in the portfolio is

$$d\Pi = r\Pi dt.$$

If it were the case that $d\Pi \neq r\Pi dt$ then one could make a risk-free profit by either borrowing Π from the bank and investing in the portfolio ($d\Pi > r\Pi dt$), or shorting the portfolio and investing the money in the bank ($d\Pi < r\Pi dt$). On replacing Π by its definition, equation (11), equation (15) is now

$$r\left(V - S\frac{\partial V}{\partial S} \right) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{16}$$

On dividing equation (16) by dt one obtains

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{17}$$

which is the Nobel prize winning Black-Scholes partial differential equation.
Remarks:

- This equation defines the price of **any** derivative claim on an underlying asset which follows geometric Brownian motion. The boundary conditions will determine which type of derivative we are evaluating.

- This is a backwards parabolic partial differential equation, a class of equations about which a lot more will be said below.

- Notice that by setting up the portfolio Π using what is known as the *Delta Hedge* the Black Scholes equation does not depend on the drift term μ in any way. The only parameter which needs to be empirically estimated is σ.

- The Delta (Δ) which is the rate of change of the derivative with respect to the underlying asset is a very important value.

- The linear operator

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r$$

  is a measure of the difference between the return on the hedged portfolio (Π) which are the first two terms (see equation (15)) and the return on a bank deposit which are the last two terms. For a European option these will be the same, though they are not necessarily for an American option.

- For many types of options it is not possible to obtain closed-form analytic values but more often than not numerical procedures must be employed. In this lecture course, though, emphasis will remain on analytic solutions.

4.3 Formulating the mathematical problem

4.3.1 Classifying the PDE

For there to be no arbitrage, the option value obtained from the Black-Scholes PDE must provide a unique option price. Later it will be shown that, given suitable boundary conditions, this is indeed the case. First, in order to determine the type of boundary conditions required it is necessary to find out some general information about the PDE itself.

We know that in general a PDE with solution $u(x,t)$ of the form

$$au_{xx} + bu_{xt} + cu_{tt} + du_x + eu_t + fu = g \tag{18}$$

is classified depending on the sign of $b^2 - 4ac$ as follows:

- If $b^2 - 4ac < 0$ then the equation is **elliptic**.
- If $b^2 - 4ac = 0$ then the equation is **parabolic**.
- If $b^2 - 4ac > 0$ then the equation is **hyperbolic**.

The most commonly seen parabolic equation is the diffusion or heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

which typically models the evolution of heat along a bar. As they are second order in x and only first order in t parabolic equations usually require two boundary conditions in x (or S in the Black-Scholes case) and just the one in t. It is important to notice here that in the heat conduction equation the $\partial u/\partial t$ term is of a different sign from that in the Black-Scholes equation (17). This is because the heat conduction equation is a *forwards* parabolic equation whilst the Black-Scholes equation is *backwards* parabolic. The difference between the two types is that *forwards* equations require *initial* conditions, whilst *backwards* equations require *final* conditions.

Note how these requirements are consistent with the individual nature of the problems. When valuing options, we know the value at expiry (or the final time) and so it makes sense that this problem gives rise to a backwards parabolic type. The heat conduction (or diffusion) equation requires a known distribution of heat on a bar (or equivalent system) at $t = 0$ and then models how the heat distribution evolves as time moves forwards. As such the system requires initial conditions - thus is a forwards parabolic type.

It is essential to always solve parabolic equations *'in the correct direction'*.

4.3.2 Characteristics

The classification of PDEs in the above section is closely related to the notion of **characteristics**. Characteristics are families of curves along which information moves or across which discontinuities may occur. The trick is to attempt to write the derivative terms in the PDE in terms of directional derivatives reducing the equation to one which behaves like an ODE along these characteristic curves.

Definition (Characteristic curve) A curve Γ is a characteristic for a general second order PDE if, for a general PDE in x and t,

$$\frac{\partial t}{\partial x} - \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = 0$$

along Γ.

Clearly the value of $b^2 - 4ac$ will be important in determining the characteristic curves. In the parabolic case there is just one real valued solution giving

$$\frac{\partial t}{\partial x} = \frac{b}{2a}.$$

In the case of the heat conduction equation where $b = 0$ then this reduces to

$$\frac{\partial t}{\partial x} = 0$$

giving characteristic curves along $t = C$ where C is a constant.

4.3.3 Boundary conditions for the Black-Scholes equation

Returning to the Black-Scholes equation, for each particular type of option we will require the following boundary conditions:

$$\begin{aligned} V(S,t) &= V_a(t) \quad \text{on} \quad S = a \\ V(S,t) &= V_b(t) \quad \text{on} \quad S = b \\ V(S,t) &= V_T(S) \quad \text{on} \quad t = T \end{aligned}$$

where $V_a(t)$ and $V_b(t)$ are known functions of time and $V_T(S)$ is, correspondingly, a known function of the underlying asset price. To demonstrate how to do this for different types of options we'll consider three cases: the standard European call and put options and a cash-or-nothing call option.

European call option, C(S,t):

The most straightforward of the conditions to determine is the final condition $C(S, t = T)$ as this is the known payoff for the call option, $(\max(S - X, 0))$, hence

$$C(S,T) = \max(S - X, 0). \tag{19}$$

The conditions for specific values of S are also reasonably straightforward. Note that from the process followed by S, namely

$$dS = \mu S dt + \sigma S dW$$

if $S = 0$ then $dS = 0$ and , hence, the underlying asset remains at 0 from then on. Hence for a call option, however small the strike price X is, this scenario will always result in the option being worthless, hence

$$C(0,t) = 0 \tag{20}$$

For large S the situation is not as clear and there are three standard conventions (of which two are provided here for brevity). As $S \to \infty$ then clearly the call option is more and more likely to be exercised and in comparison to the size of S, X will be small and so one can simply use

$$C(S,t) \to S \quad \text{as} \quad S \to \infty.$$

However, the S boundary conditions are more important when dealing with numerical procedures where a large, but finite, limit is put on S ($S_{\max}$ say). In which case, more accurate conditions are required. One possibility is to assume that the option will be exercised at expiry, receiving S plus whatever else contributes to the option's value as time moves backwards. In this way write the option price for a particular high value of S as

$$C(S,t) = S + f(t)$$

on substituting into the Black-Scholes equation (17) we're left with

$$\begin{aligned} \tfrac{df}{dt} + rS + -r(S + f(t)) &= 0 \\ \tfrac{df}{dt} &= rf(t) \end{aligned}$$

which on solving gives

$$f(t) = Ae^{rt}$$

substituting in the known time constraint from (19) we get

$$A = -Xe^{-rT}$$

and so the boundary condition for large S is

$$C(S,t) \to S - Xe^{-r(T-t)} \quad \text{as} \quad S \to \infty. \tag{21}$$

European put option, $P(S,t)$:

The case for a put option is far more straightforward. Again determining the final condition is trivial as a result of the discussion in Chapter 1, so we have

$$P(S,T) = \max(X - S, 0). \tag{22}$$

The conditions for particular values of S are extensions of the above arguments for calls, only more routine. When $S = 0$ at a particular time then by the nature of the underlying process then it will stay at 0 until expiry. Hence the put option will definitely be exercised and thus worth $X - 0 = X$ at expiry. A guaranteed amount of money, in this case X, to be received at time T is worth $Xe^{-r(T-t)}$ at time t and hence

$$P(0,t) = Xe^{-r(T-t)} \tag{23}$$

As S becomes very large then the put options will certainly not be exercised as S will be much larger than the exercise price X and so

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty. \tag{24}$$

As before the most important conditions are the final ones, but the other conditions are essential for numerical schemes as well as giving us more information about the option prices.

Cash-or-nothing/binary options:

Cash-or-nothing call (put) options (denoted $CC(S,t)$ or $CP(S,t)$) are options where, at expiry, if the underlying asset price is above (below) a certain strike price, X, then the holder receives a pre-designated cash amount A, whereas if it is below (above) this amount the holder receives nothing. Hence at expiry, $t = T$, the final condition for a cash-or-nothing call is

$$CC(S,T) = A\mathcal{H}(S - X)$$

where $\mathcal{H}(.)$ is known as the **Heaviside function**. The Heaviside function is defined as follows

$$\mathcal{H}(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

and will be important when solving PDEs later in the course. Cash-or-nothing options are a special type of option in that their payoff is completely discontinuous yet it is still possible to find an option value for them.

4.4 Analytic solutions to the Black-Scholes equation

The next chapter of the course will deal with solving the heat conduction or diffusion equation and how to adapt these techniques to solve the Black-Scholes equation for some standard option pricing problems. Before doing that we will study the analytic solutions to the valuation problems and a few more key features of options.

The **Black-Scholes formulae** for the price of European call and put options are as follows:

$$C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2) \tag{25}$$

$$P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{26}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/X) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/X) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{27}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds \tag{28}$$

which we recognise as the cumulative distribution function for a Normal distribution. Note that these expressions satisfy the put call parity and so by calculating one it is routine to calculate the other, also note that the boundary conditions at $S = 0$ and $S \to \infty$ are satisfied.

For those students interested in probability it may be worth noting that $N(d_2)$ is the probability that the option will be exercised, i.e. $S > X$ at expiry. $SN(d_1)$ is the current value of a variable that equals S_T at $t = T$ *if* $S_T > X$ and is zero otherwise.

So, what does a graph of underlying asset against option price looks like as time moves backwards from expiry? As one would expect from a PDE which is a close relative of the diffusion equation, the payoff function $\max(S - X, 0)$ gradually diffuses out as time moves backwards. The same is also true for a cash or nothing option even though the payoff is in fact discontinuous.

Example

The price of an asset (today) is £5. Find the value of a put and a call option, both with an exercise price of £6, and both with expiration dates in 9 months time. The risk-free interest rate is 3% per annum (fixed) and the volatility (constant) is 10% per $(\text{annum})^{\frac{1}{2}}$.

Solution

$r = .03$, $T - t = 0.75$, $\sigma = .1$, $S = 5$, $X = 6$.

Using the formulae.$d_1 = -1.8021$, $d_2 = -1.8888$

Then

$$\begin{aligned} N(d_1) &= N(-1.8021) = N(-1.80) - .21[N(-1.80) - N(-1.81)] \\ &= 0.0359 - .21 \times (0.0359 - 0.0351) \\ &= 0.0357 \end{aligned}$$

Similarly $N(d_2) = .0295$

Leads to $C = .0060$.

Put can be calculated similarly - but best to use put-call parity:

$$P = C - S + Xe^{-r(T-t)}.$$

and this leads to $P = 0.8725$.

4.5 Delta hedging and the other hedge parameters

A tedious, yet straightforward, calculation (see example sheet 6) will show that using the known expressions for the values of call and put options, that they have the following Δ's

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1)$$

$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1$$

What does this mean? During the lifetime of the option Δ varies between 0 for out of the money calls (puts) and 1 (-1) for in the money calls (puts) and very close to T there is in fact a step function between these two extremes.

The Δ simply approximates the rate of change of the option price wrt the underlying asset and so any slight movement in the option price value will be offset by a roughly equivalent movement in Δ of the underlying. Clearly the portfolio will have to be rebalanced as regularly as possible to have a perfect hedge. In practise the number of times a portfolio can be hedged will be limited by transaction costs.

For example, looking at the graph for the value of a cash-or-nothing call option we immediately see a problem with the delta-hedging strategy underlying the Black-Scholes analysis. If Δ is $\partial C/\partial S$ then as $t \to T$ then the Δ ranges from 0 away from $S = X$ to approaching ∞ close to $S = X$. Thus as the underlying asset price moves, huge amounts of the underlying will have to be bought and sold to keep the portfolio properly hedged.

There are ways of hedging away other risks, not just those to do with the movement of the asset price. There are hedge parameters (also known as, somewhat loosely, as *The Greeks*) for each of the principle parameters in the Black-Scholes model, namely:

- The sensitivity to the decay of time of any option V is known as the **theta** and is defined as
$$\Theta = \frac{\partial V}{\partial t}$$
- The sensitivity to the volatility is known as the **vega** and is defined as
$$\mathcal{V} = \frac{\partial V}{\partial \sigma}$$
- The sensitivity to interest rates is known as **rho** and, unsurprisingly to be
$$\rho = \frac{\partial V}{\partial r}$$
- Finally, the sensitivity of the Δ to the underlying asset is known as **gamma** and is defined as follows
$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$

Often these hedge parameters are used to see what would happen if there was a small change in one of the parameters, this is important as both r and σ are not fixed or even time dependent in practice.

4.6 Implied volatility

One of the most important parameters, and the only one which is very difficult to know for definite is the volatility, σ. There are several conventions for calculating the volatility of an underlying asset. One would perhaps assume that the best way is to look at the volatility of past returns and use this as a decent guess as to what would happen in the future. However, another way is to assume that the Black-Scholes analysis is correct and use the market prices for options to back-out the volatility, using a suitable iterative procedure such as Newton-Raphson, the only unknown being σ itself.

If one attempts this they will see a problem with the volatility. Depending on how far in or out of the money the option is the volatility may well not be constant for a given r, S, and t. So, not only is it dependent on time but also on the exercise and asset prices. Such a result is often termed the *volatility smile* although many other shapes can be observed depending on the market conditions such as a *frown*, *wry smile* etc. This is another example of the faults in the Black-Scholes model.