

NON-LOCAL MEAN FIELD DYNAMO THEORY AND MAGNETIC FRONTS IN GALAXIES

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In this article we address the problem of magnetic field generation and front propagation in turbulent electrically conducting fluids, involving the velocity with finite correlations times and small magnetic diffusivity. We suggest an integro-differential dynamo equation which involves the non-local terms for both the dynamo source α -term and turbulent transport of the mean magnetic field. We derive a set of formulas which allows us to determine the rate of magnetic front propagation in galaxies valid for arbitrary memory kernels. We illustrate the general results through the use of the exponential correlation functions for memory kernels and ‘no- z ’ approximation. We find that the memory effects have strong influence on for both the growth rate and propagation speed. We perform numerical simulations of exterior front speed, and find that the transport memory essentially decreases the propagation rate.

Keywords: Memory effects; Dynamo; Front propagation

1 INTRODUCTION

An important area of research in dynamo theory is the determination of the speed at which magnetic fronts propagate in a turbulent electrically conducting fluid (see Ruzmaikin *et al.*, 1988; Moss *et al.*, 1998, 2000; Petrov *et al.*, 2001). This problem is usually studied on the basis of a mean-field dynamo equation for a large scale magnetic field $\mathbf{B}(\mathbf{x}, t)$ (see, e.g. Moffatt, 1978; Krause and Rädler, 1980; Zeldovich *et al.*, 1983), namely

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) + \beta \Delta \mathbf{B} + \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (1)$$

where \mathbf{u} is the mean velocity field, α is the coefficient describing the α -effect, and β is the turbulent magnetic diffusivity. The mean-field dynamo equation (1) emerges from an asymptotic analysis of Maxwell’s equations for a electrically conducting fluid.

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This analysis exploits the assumption of two separated scales for the turbulent velocity field which varies only on the large integral length scale and the small scale. The correlation times of a random velocity are assumed to be zero. Despite these facts, the mean field dynamo equation has often been used for a turbulent flow involving a continuous range of both spatial and temporal scales. Finite correlation time effects in the turbulent dynamo have recently been studied by Kleecorin *et al.* (2002).

It turns out that Eq. (1), in the thin-disk approximation, can be reduced to the classical Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation for the azimuthal magnetic field (see Ruzmaikin *et al.*, 1988; Moss *et al.*, 1998). It is well known that the FKPP equation is the simplest model describing the propagation of fronts into an unstable state (see Murray, 1989). It has been found that the minimal propagation speed for a magnetic front is $u = 2\sqrt{\gamma\beta}$, where γ is the linear growth rate for the mean magnetic field. However, from a physical point of view, the FKPP equation has a disadvantage that leads to an overestimation of the minimal speed u . The reason for this is that the diffusion term $\beta\Delta\mathbf{B}$ gives rise to an infinite speed of propagation: the solution $\mathbf{B}(\mathbf{x}, t)$ of (1) with a point source at $\mathbf{x} = \mathbf{0}$ and $t = 0$ is non-zero no matter how small t and how large x become. The reason for this anomaly is due to the fact that the correlation functions of the turbulent velocity field are assumed to be a delta correlated in time. One way to overcome this problem is to modify the transport process based on the turbulent diffusion approximation by taking into account memory effects. It is our intention to introduce the mean-field dynamo theory involving the transport processes with finite velocity correlations times. In what follows we show that the simple formula $u = 2\sqrt{\gamma\beta}$ overestimates the propagation rate of exterior magnetic front. A detailed discussion of advantages and shortcoming of the FKPP equation can be found in the reviews by Hädeler (1998) and Fort and Méndez (2002).

Recently, Fedotov *et al.* (2002) have developed a phenomenological dynamo theory involving turbulent flow with finite correlation times. A fundamental problem in turbulent dynamo theory is the prediction of the rate γ at which the turbulent flow generates a magnetic field. It has been shown that finite correlations and corresponding memory effects can drastically change the dynamo growth rate. In what follows we will be concerned with the question: how do memory effects influence magnetic front propagation?

The non-local mean field dynamo equation which we shall study is a generalization of (1) given by

$$\begin{aligned} \frac{\partial\mathbf{B}}{\partial t} = & \int_0^t \nabla \times \left[\int F_\alpha(\mathbf{x} - \mathbf{y}, t - s) \mathbf{B}(\mathbf{y}, s) d\mathbf{y} \right] ds \\ & - \int_0^t \nabla \times \left[\int F_\beta(\mathbf{x} - \mathbf{y}, t - s) \nabla \times \mathbf{B}(\mathbf{y}, s) d\mathbf{y} \right] ds + \nabla \times (\mathbf{u} \times \mathbf{B}) + v_m \Delta \mathbf{B}, \end{aligned} \quad (2)$$

where $F_\alpha(\mathbf{x} - \mathbf{y}, t - s)$ and $F_\beta(\mathbf{x} - \mathbf{y}, t - s)$ are the kernels describing the memory and long range interaction effects of the turbulent flow with a continuous range of spatial and temporal scales, $v_m = c^2/4\pi\sigma$ is the magnetic diffusivity and σ is the electric conductivity. The phenomenological derivation of (2) and a discussion of F_α and F_β are given in Section 2. It should be noted that the local dynamo equation (1) can be derived from (2) under the approximations $F_\alpha(\mathbf{x} - \mathbf{y}, t - s) = \alpha\delta(\mathbf{x} - \mathbf{y})\delta(t - s)$, $F_\beta(\mathbf{x} - \mathbf{y}, t - s) = \beta\delta(\mathbf{x} - \mathbf{y})\delta(t - s)$, and $\beta \gg v_m$. In this article we shall only consider

the kinematic aspect of the problem, when the back reaction of the magnetic field on the turbulent flow and the dependencies of F_α and F_β on magnetic field \mathbf{B} are neglected. Our formulation and emphasis are primarily motivated by astrophysical applications, including the generation and propagation of magnetic fields in disk galaxies (see Beck *et al.*, 1996).

2 MEAN FIELD EQUATION WITH MEMORY AND LONG RANGE INTERACTIONS

The aim of this section is to give a heuristic derivation of the mean field equation (2) involving integrals over space and time. The starting point is the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathcal{E} + \nabla \times (\mathbf{u} \times \mathbf{B}) + \nu_m \Delta \mathbf{B}, \quad (3)$$

where \mathcal{E} is the turbulent electromotive force

$$\mathcal{E} = \langle \mathbf{u}' \times \mathbf{B}' \rangle. \quad (4)$$

Here primes denote the turbulent fluctuations of the corresponding quantities, and the angular brackets denote an ensemble averaging. The main closure problem here is to express \mathcal{E} in terms of the mean magnetic field \mathbf{B} . Let us find this expression by using an approximate equation for the fluctuations of the magnetic field \mathbf{B}' :

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{u}' \times \mathbf{B}) + \nu_m \Delta \mathbf{B}'. \quad (5)$$

It should be noted that we have omitted several terms in this equation: $\nabla \times (\mathbf{u} \times \mathbf{B}' + \mathbf{u}' \times \mathbf{B} - \langle \mathbf{u}' \times \mathbf{B}' \rangle)$. This approximation is often adopted and the detailed discussions can be found in Section 7.5 of Moffatt (1978) and Section 3.6 of Krause and Rädler (1980). By solving this equation with a zero initial condition, one can get the following expression for \mathbf{B}' as a functional of \mathbf{u}' and \mathbf{B}

$$\mathbf{B}'(\mathbf{x}, t) = \int_0^t \int G(\mathbf{x} - \mathbf{y}, t - s) \nabla \times [\mathbf{u}'(\mathbf{y}, s) \times \mathbf{B}(\mathbf{y}, s)] d\mathbf{y} ds, \quad (6)$$

where $G(\mathbf{x} - \mathbf{y}, t - s)$ is the three-dimensional Green's function for the diffusion equation with appropriate boundary conditions. Substitution of (6) into (4) gives

$$\mathcal{E}(\mathbf{x}, t) = \int_0^t \int G(\mathbf{x} - \mathbf{y}, t - s) \langle \mathbf{u}'(\mathbf{x}, t) \mathbf{x} \nabla \times [\mathbf{u}'(\mathbf{y}, s) \times \mathbf{B}(\mathbf{y}, s)] \rangle d\mathbf{y} ds. \quad (7)$$

It is well known that the turbulent electromotive force \mathcal{E} can be regarded as a linear functional of \mathbf{B} and $\nabla \times \mathbf{B}$ (see Krause and Rädler, 1980). The general expression for such a functional can be written as

$$\mathcal{E}(\mathbf{x}, t) = \int_0^t \int F_\alpha(\mathbf{x} - \mathbf{y}, t - s) \mathbf{B}(\mathbf{y}, s) d\mathbf{y} ds - \int_0^t \int F_\beta(\mathbf{x} - \mathbf{y}, t - s) \nabla \times \mathbf{B}(\mathbf{y}, s) d\mathbf{y} ds. \quad (8)$$

To find the explicit expressions for the kernels $F_\alpha(\mathbf{x} - \mathbf{y}, t - s)$ and $F_\beta(\mathbf{x} - \mathbf{y}, t - s)$ it is necessary to specify statistical properties of random velocity field \mathbf{u}' . We assume that the turbulent flow is homogeneous and isotropic, then by using (7) and (8) one can find

$$F_\alpha(\mathbf{x} - \mathbf{y}, t - s) = -\frac{1}{3}G(\mathbf{x} - \mathbf{y}, t - s)\langle \mathbf{u}'(\mathbf{x}, t) \cdot \nabla \times \mathbf{u}'(\mathbf{y}, s) \rangle, \quad (9)$$

$$F_\beta(\mathbf{x} - \mathbf{y}, t - s) = \frac{1}{3}G(\mathbf{x} - \mathbf{y}, t - s)\langle \mathbf{u}'(\mathbf{x}, t) \cdot \mathbf{u}'(\mathbf{y}, s) \rangle. \quad (10)$$

The details of derivation of (9), and (10) can be found in Section 3.6 of Krause and Rädler (1980). We see that \mathcal{E} can be completely determined if we know the two-point correlation functions $\langle \mathbf{u}'(\mathbf{x}, t) \cdot \nabla \times \mathbf{u}'(\mathbf{y}, s) \rangle$ and $\langle \mathbf{u}'(\mathbf{x}, t) \cdot \mathbf{u}'(\mathbf{y}, s) \rangle$. The standard approximation of delta-correlations in time

$$\langle \mathbf{u}'(\mathbf{x}, t) \cdot \nabla \times \mathbf{u}'(\mathbf{y}, s) \rangle = -3\alpha \delta(t - s), \quad (11)$$

$$\langle \mathbf{u}'(\mathbf{x}, t) \cdot \mathbf{u}'(\mathbf{y}, s) \rangle = 3\beta \delta(t - s),$$

and the limit $\nu_m \rightarrow 0$ gives us a classical expression for the electromotive force \mathcal{E} , namely

$$\mathcal{E}(\mathbf{x}, t) = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B}, \quad (12)$$

where α and β stand for the phenomenological parameters that play a main role in dynamo theory.

Substitution of (8) into (3) gives us the integro-differential equation (2). In what follows we use this equation to give an illustration on the novel effects that can occur as a result of the non-locality of (2).

3 MAGNETIC FIELD GENERATION AND FRONT PROPAGATION

In this section we focus our attention on the limit of an infinitely large conductivity ($\sigma \rightarrow \infty$), with the implication that the magnetic diffusivity $\nu_m = c^2/4\pi\sigma$ can be neglected. This case is of special interest for many problems of physics of plasma, astro- and geophysics (see Krause and Rädler, 1980). The functions F_α and F_β appearing in the expression for the turbulent electromotive force \mathcal{E} can then be written as

$$F_\alpha(\mathbf{x} - \mathbf{y}, t - s) = \alpha(\mathbf{x})G_\alpha(t - s)\delta(\mathbf{x} - \mathbf{y}), \quad (13)$$

$$F_\beta(\mathbf{x} - \mathbf{y}, t - s) = \beta(\mathbf{x})G_\beta(t - s)\delta(\mathbf{x} - \mathbf{y}), \quad (14)$$

where G_α and G_β are the memory kernels. It should be noted that the wave analysis of the integro-differential equation (2) with the arbitrary kernels F_α and F_β describing the long range interaction and memory effects can be done according to the recent theory of Fedotov and Okuda (2002). However, we believe that for the magnetic front propagation problem it is more important to take into account memory effects rather than the long range interaction in space. It is clear from (9) and (10) that delta-correlations in space follow from the limit $\nu_m \rightarrow 0$ when the Green's function

for the diffusion equation becomes a delta-function. In astrophysics, the ratio of the magnetic diffusivity $\nu_m = c^2/4\pi\sigma$ and turbulent diffusivity β is a small parameter that might lead to insignificant corrections of propagation speed.

Let us discuss the influence of the non-local in time effects by considering an example of magnetic field generation in a thin disc. This example appears to be very useful for astrophysics, describing disc galaxies as a thin turbulent slab of the thickness $2h$ and radius R ($R \gg h$) which rotates with the angular velocity $\omega(r)$ (see Beck *et al.*, 1996). It is convenient to consider the polar cylindrical coordinates (r, φ, z) with z -axis coincident with the rotation axis. For simplicity, we neglect the effects of compressibility, diamagnetism and deviations from the axial symmetry, and assume that $\alpha = \alpha(z)$ and that β is constant. The governing equations for the mean axisymmetric magnetic field follow from (2), (13), and (14). They are

$$\begin{aligned} \frac{\partial B_r}{\partial t} = & - \int_0^t \frac{\partial}{\partial z} [\alpha(z) G_\alpha(t-s) B_\varphi] ds \\ & + \beta \int_0^t G_\beta(t-s) \left\{ \frac{\partial^2 B_r}{\partial z^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_r) \right] \right\} ds, \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{\partial B_\varphi}{\partial t} = & g B_r + \int_0^t \frac{\partial}{\partial z} [\alpha(z) G_\alpha(t-s) B_r] ds \\ & + \beta \int_0^t G_\beta(t-s) \left\{ \frac{\partial^2 B_\varphi}{\partial z^2} + \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_\varphi) \right] \right\} ds, \end{aligned} \quad (16)$$

where $g = r d\omega/dr$ is the measure of differential rotation. Here we are only interested in the B_r and B_φ components of the magnetic field \mathbf{B} , since $B_z/B_{r,\varphi} = O(h/R)$. These components obey the so-called vacuum boundary conditions. Since there are no electric currents outside the disk, for axisymmetric solutions we have $B_{r,\varphi}(t, r, z = \pm h) = 0$. The detailed discussion of this approximate boundary condition can be found in the books by Zeldovich *et al.*, p. 151 (1983) and Ruzmaikin *et al.*, p. 182 (1988).

Now we are in a position to discuss the problem of magnetic front propagation in spiral galaxies. The wavefronts are special solutions to the Eqs. (15) and (16) that travel with constant shape and speed connecting an unstable initial state and a stable final state. It has been shown (see Murray, 1989) that a sufficiently localized initial disturbance evolves asymptotically ($t \rightarrow \infty$) into a travelling monotonic wavefront. The speed u at which the front propagates into an unstable state is referred to as the selected speed. Let us assume that the dynamo excitation occurs within a certain radius $r \leq r_0$, then the magnetic front propagates into the region $r > r_0$. The growth rate γ is assumed to remain positive for $r > r_0$. This type of magnetic front is referred to as an exterior front (Moss *et al.*, 2000). Our aim is now to find the propagation rate of the magnetic front that develops after some transient period of time. Since we are interested in the long-time large-distance asymptotic limit, it is convenient to consider the case when $t \rightarrow \infty$, $r \rightarrow \infty$. The great advantage of considering the exterior front is that its propagation rate can be found from linear analysis (Murray, 1989). Because of the great difference between vertical and horizontal dimensions of the spiral galaxies, we assume that the ratio $\varepsilon = h/R$ tends to zero. This allows us to consider the asymptotic limit $r \rightarrow \infty$, which should be considered as the intermediate one. We also assume that $h/r_0 \rightarrow 0$, and, therefore, the terms proportional to $1/r^2$ and

$1/r$ can be neglected in the Eqs. (15) and (16). In this case we consider the propagation of the effectively plane magnetic front neglecting all curvature effects.

To obtain the rate of propagation we are going to apply the general technique developed by Ebert and Saarloos (2000) and Fedotov (2001). We first illustrate it using the relatively simple example of the linearized FKPP equation for the scalar field $\varphi(t, x)$, namely

$$\frac{\partial \varphi}{\partial t} = \beta \frac{\partial^2 \varphi}{\partial x^2} + \gamma \varphi. \quad (17)$$

We are looking for a solution in the exponential form

$$\varphi = \varphi_0 \exp(Et - pr). \quad (18)$$

Substituting (18) into (17) gives the equation for the effective Hamiltonian function $H(p) = E$, namely

$$H(p) = \beta p^2 + \gamma. \quad (19)$$

It follows from the general theory of wave propagation into an unstable state (see Ebert and Saarloos, 2000; Fedotov, 2001) that the propagation rate u can be determined by

$$u = \frac{H(p)}{p} = \frac{dH(p)}{dp}. \quad (20)$$

From (19) and (20) we easily find the classical result $u = 2\sqrt{\gamma\beta}$ with $p = \sqrt{\gamma/\beta}$. The momentum p can be interpreted as the inverse characteristic width of the travelling wave.

Now let us consider the system (15) and (16) and find its solution in the following form

$$B_r = b_r(z) \exp(Et - pr), \quad (21)$$

$$B_\varphi = b_\varphi(z) \exp(Et - pr). \quad (22)$$

Substitution of (21) and (22) into (15) and (16) gives an one-dimensional eigenvalue problem for $b_r(z)$ and $b_\varphi(z)$, namely

$$Eb_r = -\frac{d}{dz}[\alpha_E(z)b_\varphi] + \beta_E \frac{d^2 b_r}{dz^2} + \beta_E p^2 b_r, \quad b_r(z = \pm h) = 0, \quad (23)$$

$$Eb_\varphi = g b_r + \frac{d}{dz}[\alpha_E(z)b_r] + \beta_E \frac{d^2 b_\varphi}{dz^2} + \beta_E p^2 b_\varphi, \quad b_\varphi(z = \pm h) = 0. \quad (24)$$

Here we have introduced the parameters $\alpha_E(z)$ and β_E , namely

$$\alpha_E(z) = \alpha(z)\hat{G}_\alpha(E), \quad \beta_E = \beta\hat{G}_\beta(E) \quad (25)$$

and the Laplace transforms of the corresponding kernels $\hat{G}_\alpha(E)$ and $\hat{G}_\beta(E)$, namely

$$\hat{G}_{\alpha,\beta}(E) = \int_0^\infty G_{\alpha,\beta}(s) \exp[-Es] ds. \quad (26)$$

If we put $p = 0$ in (23) and (24), we get the classical generation equations (see Zeldovich *et al.*, 1983). When $p \neq 0$ Eqs. (23) and (24) can be rewritten in a very useful form

$$E - \beta_E p^2 = -\frac{1}{b_r} \frac{d}{dz} [\alpha_E(z) b_\varphi] + \frac{\beta_E}{b_r} \frac{d^2 b_r}{dz^2} = \gamma, \quad b_r(z = \pm h) = 0, \quad (27)$$

$$E - \beta_E p^2 = \frac{g}{b_\varphi} \frac{b_r}{dz} + \frac{1}{b_\varphi} \frac{d}{dz} [\alpha_E(z) b_r] + \beta_E \frac{d^2 b_\varphi}{dz^2} = \gamma, \quad b_\varphi(z = \pm h) = 0. \quad (28)$$

It follows that if we find the largest eigenvalue of the above problem, namely

$$\gamma = f(E), \quad (29)$$

then the Hamiltonian $H(p) = E$ corresponding to (15) and (16) can be found from the equation

$$H - \beta_E p^2 = f(H), \quad (30)$$

since $E - \beta_E p^2 = \gamma$ (see Eqs. (27) and (28))

In principle, the formulas (20), (29) allow us to determine the rate of propagation u of fronts outward from central regions for arbitrary time-correlation functions G_α and G_β , provided we know the largest eigenvalue $\gamma = f(E)$ of the problem (27), (28). The propagation rate u can then easily be found numerically.

It should be mentioned that for real galaxies the local growth rate γ decreases slowly with radius, approximately as $1/r$. It is often assumed that $\alpha \sim \omega$, where ω is the angular velocity of rotation. Since $\gamma \sim \sqrt{\alpha r d\omega/dr}$, we have $\gamma \sim \omega$ with $\omega \sim 1/r$ at large r . It appears that the average propagation rate should be lower than the prediction of our asymptotic theory with γ constant. One can also treat the radius r as a parameter. Then the position of magnetic front $r(t)$ can be found from a simple equation $dr/dt = u(r(t))$, where $u(r)$ is given by (20). It is clear that the propagation rate is not a constant anymore. An estimate of time dependence of propagation rate can be found from the following arguments. Since $u \sim \sqrt{\gamma(r)}$ with $\gamma \sim 1/r$, $dr/dt \sim r^{-1/2}(t)$ implies $r(t) \sim t^{2/3}$ and $u(t) = dr/dt \sim t^{-1/3}$ for large t .

4 EXPONENTIAL MEMORY KERNELS

Let us illustrate the above theory by using exponential forms for the kernels G_α and G_β , namely

$$G_\alpha(t-s) = \frac{1}{\tau_\alpha} \exp\left(-\frac{t-s}{\tau_\alpha}\right), \quad G_\beta(t-s) = \frac{1}{\tau_\beta} \exp\left(-\frac{t-s}{\tau_\beta}\right). \quad (31)$$

We introduce the following dimensionless variables

$$z \rightarrow z/h, \quad t \rightarrow \beta t/h^2, \quad \alpha \rightarrow \alpha_0 \alpha(z), \quad (32)$$

and parameters

$$T_\alpha = \alpha_0 \tau_\alpha/h, \quad T_\beta = \beta \tau_\beta/h^2, \quad R_\alpha = \alpha_0 h/\beta, \quad R_\omega = gh^2/\beta. \quad (33)$$

Let us note that the example of the $\alpha\omega$ -dynamo corresponds to the case: $R_\alpha \ll |R_\omega|$. The field generation equations in these notation take the form

$$\gamma b_r = -\frac{R_\alpha}{1 + \gamma T_\alpha/R_\alpha} \frac{d}{dz} [\alpha(z) b_\phi] + \frac{1}{1 + \gamma T_\beta} \frac{d^2 b_r}{dz^2}, \quad (34)$$

$$\gamma b_\phi = R_\omega b_r + \frac{R_\alpha}{1 + \gamma T_\alpha/R_\alpha} \frac{d}{dz} [\alpha(z) b_r] + \frac{1}{1 + \gamma T_\beta} \frac{d^2 b_\phi}{dz^2}, \quad (35)$$

where the parameter γ stands for the dimensionless growth rate of magnetic field. In the case when $T_{\alpha,\beta} = 0$, the eigenvalue problem (34) and (35) reduces to a well-known form (see Zeldovich *et al.*, 1983). Otherwise, the eigenvalue γ becomes a function of the dimensionless correlation times T_α and T_β . By using the renormalized parameters $\tilde{\gamma} = \gamma(1 + \gamma T_\beta)$, $\tilde{R}_\omega = R_\omega(1 + \gamma T_\beta)$, $\tilde{R}_\alpha = R_\alpha(1 + \gamma T_\beta)/(1 + \gamma T_\alpha/R_\alpha)$, γ can be easily determined from the following problem

$$\left(\tilde{\gamma} + \frac{d^2}{dz^2} \right) b_r = -\tilde{R}_\alpha \frac{d(\alpha b_\phi)}{dz}, \quad \left(\tilde{\gamma} + \frac{d^2}{dz^2} \right) b_\phi = \tilde{R}_\omega b_r + \tilde{R}_\alpha \frac{d(\alpha b_r)}{dz}, \quad (36)$$

$$b_{r,\phi}(z = \pm h) = 0.$$

The eigenvalue problem (36) corresponds to the generation equations in the local mean field dynamo theory (see Zeldovich *et al.*, 1983). The difference is that the renormalized parameters $\tilde{\gamma}$, \tilde{R}_α and \tilde{R}_ω are dependent upon the growth rate γ . It is well known that the behaviour of magnetic field depends on the value of so-called dynamo numbers R_α and R_ω ($R_\alpha \sim 1 \sim 10$, $R_\omega \sim 10 - 10^3$). The asymptotics of large R_ω/R_α gives (see Zeldovich *et al.*, 1983)

$$\gamma \left(\sqrt{1 + \gamma T_\alpha/R_\alpha} \right) \simeq \sqrt{-D} \cdot \text{const} \equiv \gamma(T_{\alpha,\beta} = 0), \quad D = R_\omega R_\alpha, \quad (37)$$

where the constant coefficient has to be determined by the function $\alpha(z)$. An explicit form is given in the book by Zeldovich *et al.* (1983). In this case, the contribution from the non-locality in the β -term is weak due to the large values of D ; however, the non-locality in the dynamo α -source plays a significant role (Fig. 1) and leads to a significant decrease in the dimensionless growth rate γ .

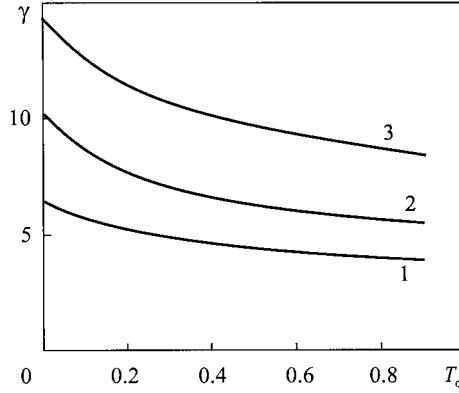


FIGURE 1 Dependence of the growth rate γ on the α -correlation time T_α ($T_\beta = 0$) for the dynamo parameters: $R_\alpha = 2$, $R_\omega = -40$ (curve 1); 2, -100 (curve 2); 4, -100 (curve 3).

5 SIMPLE ‘NO-Z’ MODEL

In this section we consider a galactic disc with a uniform semi-thickness h . To simplify the basic Eqs (15) and (16) we use the so-called ‘no- z ’ model of Moss (1995). The main idea of this model is to replace the z -derivatives by inverse power of h . The ‘no- z ’ model has been widely used to study galactic dynamos, and appears to be adequate to most observations of the magnetic field generation in disc-like galaxies (see Beck *et al.*, 1996). Taking into account the memory effects, the dynamo equations for the radial B_r and the azimuthal B_φ components of an axisymmetric magnetic field in the ‘no- z ’ approximation can be written as

$$\frac{\partial B_r}{\partial t} = -\frac{\alpha_0}{\tau_\alpha h} \int_0^t \exp\left(-\frac{t-s}{\tau_\alpha}\right) B_\varphi ds + \frac{\beta}{\tau_\beta} \int_0^t \exp\left(-\frac{t-s}{\tau_\beta}\right) \left[\Delta_r B_r - \frac{B_r}{h^2}\right] ds, \quad (38)$$

$$\frac{\partial B_\varphi}{\partial t} = g B_r + \frac{\alpha_0}{\tau_\alpha h} \int_0^t \exp\left(-\frac{t-s}{\tau_\alpha}\right) B_r ds + \frac{\beta}{\tau_\beta} \int_0^t \exp\left(-\frac{t-s}{\tau_\beta}\right) \left[\Delta_r B_\varphi - \frac{B_\varphi}{h^2}\right] ds. \quad (39)$$

In the standard thin disk approximation, the B_z component is small ($B_z/B_r = O(h/R)$) and can therefore be ignored. When the correlation times τ_α and τ_β are zero, these equations coincide with those in Moss *et al.* (1998). By using the same notation as in Eqs. (34) and (35) and setting $r \rightarrow r/h$, one can rewrite Eqs. (38) and (39) in the following non-dimensional form

$$\frac{\partial B_r}{\partial t} = -\frac{R_\alpha}{T_\alpha} \int_0^t \exp\left[-\frac{R_\alpha}{T_\alpha}(t-s)\right] B_\varphi ds + \frac{1}{T_\beta} \int_0^t \exp\left(-\frac{t-s}{T_\beta}\right) [\Delta_r B_r - B_r] ds, \quad (40)$$

$$\frac{\partial B_\varphi}{\partial t} = R_\omega B_r + \frac{R_\alpha}{T_\alpha} \int_0^t \exp\left[-\frac{R_\alpha}{T_\alpha}(t-s)\right] B_r ds + \frac{1}{T_\beta} \int_0^t \exp\left(-\frac{t-s}{T_\beta}\right) [\Delta_r B_\varphi - B_\varphi] ds. \quad (41)$$

Looking for a solution of the form $B_r = b_r \exp(\gamma t)$, $B_\varphi = b_\varphi \exp(\gamma t)$, one can find the characteristic equation for the dimensionless growth rate γ :

$$\left[\gamma + \frac{1}{1 + \gamma T_\beta} \right]^2 = - \frac{R_\alpha}{(1 + \gamma T_\alpha / R_\alpha)} \left(R_\omega + \frac{R_\alpha}{(1 + \gamma T_\alpha / R_\alpha)} \right). \quad (42)$$

In the limit $T_\alpha \rightarrow 0$ and $T_\beta \rightarrow 0$ we find

$$\gamma = -1 + \sqrt{-R_\alpha(R_\omega + R_\alpha)}. \quad (43)$$

For the $\alpha\omega$ -dynamo ($R_\alpha \ll |R_\omega|$) the growth rate is given by $\gamma = -1 + \sqrt{-D}$, where $D = R_\alpha R_\omega$ is the dynamo number. It should be noted that the critical value $D_{cr} = -1$ does not depend on memory effects. The dependency of γ on the relaxation time, for the same system parameters as in Fig. 1, are shown in Fig. 2. A comparison of the results presented in Figs. 1 and 2 confirms that the ‘no- z ’ model is indeed a good approximation for the large dynamo numbers. However, for dynamo numbers in the neighbourhood of the critical value for full model $D_{cr} = -\pi^4/16$, at which the generation of magnetic field occurs, the results of the full model and those of the ‘no- z ’ model are different. This discrepancy can be explained by the fact that ‘no- z ’ model gives us an averaged description along the z -direction. It should be noted the discrepancy near $D = D_{cr}$ can be reduced by replacing $\partial^2 B / \partial z^2$ by $-(\pi^2/4h^2)B$ rather than $-(1/h^2)B$ (Phillips, 2000).

Let us now analyze the exterior front propagation in the disk plane (Moss *et al.*, 2000). We assume that at the moment $t = 0$ the magnetic field \mathbf{B} is

$$\mathbf{B} = 0, \quad \text{if } r > r_0; \quad \mathbf{B} = \mathbf{B}_0, \quad \text{if } r \leq r_0. \quad (44)$$

We expect that this initial compact distribution develops into a travelling wave as $t \rightarrow \infty$. Let us find the solution of the system (38) and (39) in the form

$$B_r = b_r \exp(Et - pr), \quad (45)$$

$$B_\varphi = b_\varphi \exp(Et - pr). \quad (46)$$

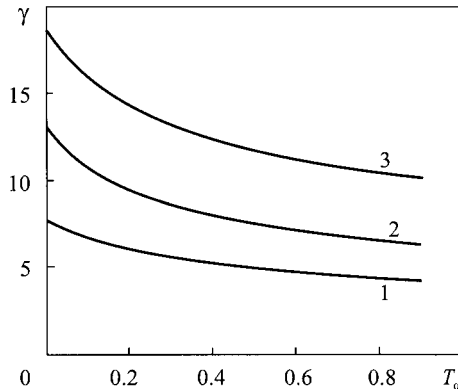


FIGURE 2 The same as on Fig. 1 for the growth rate γ calculated by ‘no- z ’ model.

The dispersion relation determining the energy $E = H(p)$ as a function of the momentum p is

$$\left[H + \frac{1 - p^2}{1 + HT_\beta} \right]^2 = - \frac{R_\alpha}{(1 + HT_\alpha/R_\alpha)} \left(R_\omega + \frac{R_\alpha}{(1 + HT_\alpha/R_\alpha)} \right). \quad (47)$$

Again one can use (47) and the formulas (20) to determine the propagation rate u .

It is convenient to consider the case when $T_\alpha = 0$, but $T_\beta \neq 0$ ($\tau_\alpha = 0$, $\tau_\beta \neq 0$). It turns out that in this case the integro-differential Eq. (2) with zero mean velocity ($\mathbf{u} = 0$) can be rewritten as a hyperbolic equation of telegraph type. By using (13), (14), and the exponential functions (31) with $\tau_\alpha = 0$, $\tau_\beta \neq 0$, and $v_m = 0$, one can write (2) in the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) + \frac{\beta}{\tau_\beta} \int_0^t \exp\left(-\frac{t-s}{\tau_\beta}\right) \Delta \mathbf{B}(\mathbf{x}, s) ds. \quad (48)$$

Differentiation with respect to time t gives

$$\tau_\beta \frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{\partial}{\partial t} [\mathbf{B} - \tau_\beta \nabla \times (\alpha \mathbf{B})] = \nabla \times (\alpha \mathbf{B}) + \beta \Delta \mathbf{B}. \quad (49)$$

It is well known that this equation, unlike standard parabolic equations, corresponds to a transport phenomena with a finite velocity of propagation equal to $\sqrt{\beta/\tau_\beta}$. Therefore, the integral non-local model (2) takes into account the fact that the propagation of a magnetic field is characterized by a finite maximal velocity, which is determined by the large turbulent eddies.

When $T_\alpha = 0$, $T_\beta \neq 0$, the positive solution of the dispersion relation (47) is

$$H(p) = \frac{1}{2T_\beta} \left[-(1 - wT_\beta) + \sqrt{(1 + wT_\beta)^2 - 4T_\beta + 4T_\beta p^2} \right], \quad (50)$$

$$w = \sqrt{-R_\alpha(R_\omega + R_\alpha)} = \sqrt{-\frac{\alpha_0 h^2}{\beta^2} (gh + \alpha_0)}.$$

Using (20) one can find the dimensionless magnetic front propagation rate

$$u = 2 \sqrt{\frac{w - 1}{(1 + wT_\beta)^2 - 4T_\beta}} \quad (51)$$

An interesting feature of this formula is that if $w = 1$ the propagation rate becomes zero. This is a result of the zero boundary conditions that lead to the phenomenon of propagation failure: when $w < 1$, the wave ceases to exist. When $T_\beta = 0$ the propagation rate is $u = 2\sqrt{w-1}$. When T_β increases, u decreases monotonically if $w > 2$ and has the minimal value $u_{\min} = \sqrt{1/T_\beta}$, when $wT_\beta = 1$. It should be noted that this velocity coincides with the velocity of hyperbolic waves in the Eq. (49) without the α -source. When $1 < w < 2$ the front velocity first increases with T_β , then reaches the maximum value $v_{\max} = 2w$, when $T_\beta = 2w^{-2}(2-w)$, then u decreases to the u_{\min} .

Some results, illustrating the dependencies of u on T_β , are shown in Fig. 3. In the limit $T_\beta \rightarrow 0$ and $|R_\omega| \gg R_\alpha$ the formula (51) reduce to the form given by Petrov *et al.* (2001). This corresponds to the classical FKPP result: the speed of the exterior front is $2\sqrt{\gamma_G \beta}$, where γ_G is the dimensional local growth rate of the dynamo.

Unfortunately there is no analytical solution of the dispersion relation (47) if $T_\alpha \neq 0$. The velocity u as function of T_α is presented in Fig. 4. If we compare Figs. 3 and 4, we can see that T_β provides a greater influence on the velocity u than T_α . It is an inverse situation to that of studying the dependency of the growth rate γ on the relaxation time. Therefore, memory effects do not lead to a simple renormalization of the phenomenological parameters α and β . Figure 5 illustrates the dependence of u on $T = T_\alpha = T_\beta$.

To assess the significance of the memory effects it is useful to estimate the magnitudes of T_α and T_β . First let us find the typical value of the parameter $T_\beta = \beta \tau_\beta / h^2$, where h is the half-thickness of the gaseous disc ($h \simeq 400$ pc). The turbulent magnetic diffusivity is given by $\beta \simeq lv$, where v is the typical velocity of turbulent eddy, and l is the typical size ($l \simeq 100 \div 200$ pc). The relaxation time τ_β can be regarded as the typical turnover time, and, therefore, $\tau_\beta = l/v$. This gives the estimate $T_\beta \simeq (l/h)^2$. It is clear that T_β is

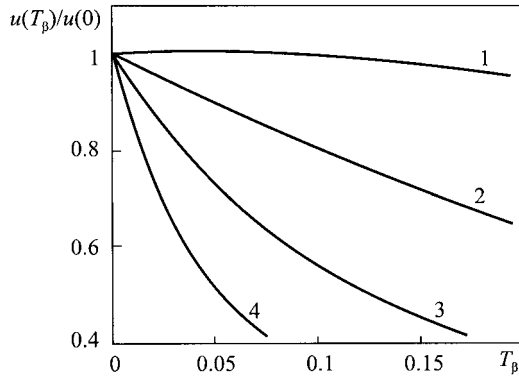


FIGURE 3 Relative front velocity $u(T_\beta)/u(T_\beta = 0)$ vs turbulent transport correlation time T_β ($T_\alpha = 0$) for the dynamo parameters: $R_\alpha = 1$, $R_\omega = -4.5$ (curve 1); 1, -20 (curve 2); 2, -40 (curve 3); 4, -100 (curve 4).

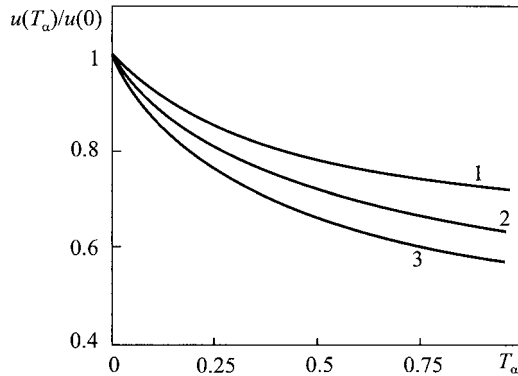


FIGURE 4 Relative front velocity $u(T_\alpha)/u(T_\alpha = 0)$ vs α -correlation time T_α ($T_\beta = 0$) for the dynamo parameters: $R_\alpha = 2$, $R_\omega = -40$ (curve 1); 4, -40 (curve 2); 4, -100 (curve 3).

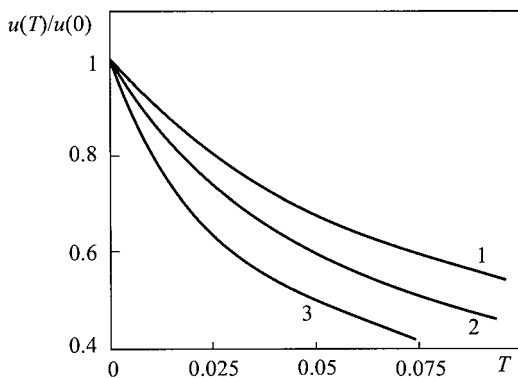


FIGURE 5 Relative front velocity $u(T)/u(T=0)$ vs correlation times $T = T_\alpha = T_\beta$ for the dynamo parameters: $R_\alpha = 2$, $R_\omega = -40$ (curve 1); 4, -40 (curve 2); 4, -100 (curve 3).

bounded by one, but is not necessarily a small parameter. Let us now estimate an important parameter $T_\alpha = \alpha_0 \tau_\alpha / h$ describing the memory of α -effect. The typical values of α_0 are $1 \div 10 \text{ km s}^{-1}$ (see Ruzmaikin *et al.*, 1988), that is, α_0 can be the same order as the turbulent eddy velocity v ($\alpha_0 \leq v$). If we assume that $\tau_\alpha \simeq \tau_\beta = l/v$ then the estimate for T_α is given by $T_\alpha \leq l/h \leq 1$ ($l/h \sim 0.25 \div 0.5$ is plausible for general orientation).

CONCLUSIONS

In this article we have discussed the problem of the magnetic field generation and front propagation in electrically conducting fluids involving turbulent velocity with finite correlation times and a small magnetic diffusivity. The main motivation for this has been that the diffusion approximation for the turbulent transport process admits an infinite speed of transport propagation, and as such leads to an overestimation of the magnetic front propagation rate. We have suggested an integro-differential dynamo equation involving non-local spatial and temporal terms for the dynamo α -source and turbulent transport of the mean magnetic field. We have derived a set of formulas which allows us to determine the rate of magnetic exterior front propagation valid for arbitrary memory kernels. We have given an illustration of this theory by using exponential correlation functions and a ‘no- z ’ approximation. We have performed a numerical simulation of the front speed and found that the transport memory significantly decreases the propagation rate.

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