TWO AND A HALF HOURS

THE UNIVERSITY OF MANCHESTER

CALCULUS AND VECTORS

DATE: 12 January 2016

TIME: 14:00 - 16:30

Answer ALL FIFTEEN questions

Electronic calculators and formula tables are not permitted.

No prepared notes of any kind are to be brought into the examination room.

This examination makes up 75% of the overall assessment for this course unit.

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1. Sketch graphs of the following real-valued functions of x satisfying

(a) $f(x) = e^{-|x-1|} - 1$; (b) $f(x) = \ln(4- \mid x \mid)$. [4]

2. Sketch in the complex plane the region where complex values of z satisfy $2 <\mid 2z - 2 + 2i \mid< 4$. [2]

3. By using implicit differentiation, find the derivative of the inverse trigonometric function

$$\tan^{-1}(x).$$

[4]

4. A function f is defined by

$$f(x) = \sqrt{-2x-1}.$$

(a) Find a formula for the inverse function $f^{-1}(x)$;

(b) sketch the graphs of $f^{-1}(x)$ and $f(x)$ using the same coordinate axes. [4]

5. Find an equation of the tangent line to the curve $y = e^x$ that is parallel to the line

$$x - 4y = 1.$$

[4]

6. Find the limits

(a) $\lim\limits_{x\to 1} \dfrac{1 - x + \ln x}{1 + \cos(\pi x)}$; (b) $\lim\limits_{x\to\infty} x \sin\left(\dfrac{\pi}{x}\right)$. [4]

7. For what value of k does the equation

$$e^{2x} = k\sqrt{x}$$

have exactly one solution. [5]

8. Find the area of the region bounded by the curve $y = \sin(\pi x/2)$ and the line $y = x$. [4]

9. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

[5]

10. Find the exact length of the curve $x(t) = 1 + 3t^2, \quad y(t) = 4 + 2t^3$, $0 < t < 1$. [5]

11. Find an equation of the plane that passes through the point $(6, 0, -2)$ and contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$. [6]

12. Find the directional derivative of the function $f(x, y) = x^2 e^y$ at the point $(2, 0)$ in the direction of $\vec{v} = (1, 1)$. [4]

13. Evaluate the double integral

$$\int\int_D (3x + 4y^2) dA,$$

where D is the region in the upper half-plane bounded by the curves $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. [8]

14. By using polar coordinates, or otherwise, find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$. [10]

15. Find an equation of the tangent plane to the surface

$$-3x^2 + y^2 - 2x + z = 0$$

at the point $(1, -2, 1)$. Find the symmetric equations for the normal line to this tangent plane. [6]

END OF EXAMINATION PAPER