Introduction to Financial Mathematics - 20912

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Exercise Sheet 6 - Black-Scholes Equation and Greeks

Recall that the Black-Scholes equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

has the explicit solution for the European call option:

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy, \quad d_1 = \frac{\ln(S/E) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

1. Show by substitution that the following functions are the exact solutions of the Black-Scholes equation

(a) $C(S,t) = AS$, (A is the arbitrary constant);

(b) $C(S,t) = S - Ke^{-r(T-t)}$ (the value of the **forward contract**, where $K = const$ is the delivery price)

2. Find all parameters α for which the function $C(S,t) = S^{\alpha}e^{-r(T-t)}$ is the solution of the Black-Scholes equation.

Ans: $\alpha_1 = 0$ and $\alpha_2 = 1 - \frac{2r}{\sigma^2}$.

3. Calculate the price of a three-month European call option on a stock with a strike price \$60 when the current stock price is \$80. The risk-free interest rate is 10% per annum. The volatility is 30%.

Ans: $C_0 = 21.549$

4. *The Greek Letters.*

Show that (a) (not easy!)

$$\Delta = \frac{\partial C}{\partial S} = N(d_1);$$

Hint: use the explicit solution for the European call option

(b)

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}};$$

(c)

$$\Theta = \frac{\partial C}{\partial t} = -\frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - rEe^{-r(T-t)}N(d_2).$$

(Hint: use the expressions for Δ , Γ , the Black-Scholes equation and its exact solution)