Introduction to Financial Mathematics - 20912

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Exercise Sheet 8: Solutions

1. (a) The solution of the equation $\frac{dV}{dt} = r(t)V - K(t)$ can be written as

$$V(t) = F \exp\left(-\int_t^T r(s)ds\right) + V_1(t),$$

where $V_1(t) = C(t)\exp\left(-\int_t^T r(s)ds\right)$. To find $C(t)$, let us substitute $V_1(t)$ into the equation $\frac{dV_1}{dt} = r(t)V_1 - K(t)$:

$$\frac{dC}{dt}\exp\left(-\int_t^T r(s)ds\right) + C(t)\,r(t)\exp\left(-\int_t^T r(s)ds\right) = r(t)C(t)\exp\left(-\int_t^T r(s)ds\right) - K(t).$$

It follows from here that $C(t)$ is the solution of the equation

$$\frac{dC}{dt} = -K(t)\exp\left(\int_t^T r(s)ds\right).$$

Let us solve it with the final condition $C(T) = 0$. This condition follows from $V_1(T) = 0$. Integration from t to T gives $C(T) - C(t) = -\int_t^T K(y)\exp\left(\int_y^T r(s)ds\right)dy$, therefore

$$C(t) = \int_t^T K(y)\exp\left(\int_y^T r(s)ds\right)dy$$

and

$$V(t) = \exp\left(-\int_t^T r(s)ds\right)\left[F + \int_t^T K(y)\exp\left(\int_y^T r(s)ds\right)dy\right].$$

(b) When the interest rate r is constant, we obtain that $\exp\left(-\int_t^T r(s)ds\right) = e^{-r(T-t)}$. Thus

$$V(t) = Fe^{-r(T-t)} + \int_t^T K(y)e^{-r(y-t)}dy.$$

and

$$V(0) = Fe^{-rT} + \int_0^T K(y)e^{-ry}dy.$$

2. (a) Rewriting $r(t)$ as

$$r(t) = r_2 + \frac{r_1 - r_2}{1+t}$$

one can find

$$\int_t^T r(s)ds = \int_t^T \left(r_2 + \frac{r_1 - r_2}{1+s}\right)ds = r_2(T-t) + (r_1 - r_2)\ln\left(\frac{1+T}{1+t}\right).$$

Therefore

$$V(t) = F\exp\left(-\int_t^T r(s)ds\right) = F\left(\frac{1+T}{1+t}\right)^{(r_2 - r_1)}\exp\left(-r_2(T-t)\right)$$

and

$$V(0) = F\,(1+T)^{(r_2-r_1)} \exp\left(-r_2 T\right).$$

(b) The term structure of interest rate $Y\,(0,T)$ is

$$Y(0,T) = \frac{1}{T}\int_0^T r(s)ds = r_2 + (r_1 - r_2)\,\frac{1}{T}\ln\left(1+T\right).$$

To plot the function $Y(0,T)$, you need the $\lim_{T\to 0}\frac{\ln(1+T)}{T} = 1$

3. If $K = const$, then

$$V(0) = \exp\left(-\int_0^T r(s)ds\right)\left[F + K\int_0^T \exp\left(\int_y^T r(s)ds\right)dy\right].$$

If $r(t) = r = const$, then $K\int_0^T \exp\left(r\int_y^T ds\right)dy = K\frac{-1+e^{rT}}{r}$, therefore

$$V(0) = e^{-rT}\left[F + K\frac{e^{rT}-1}{r}\right].$$