# Introduction to Financial Mathematics - 20912

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**Solutions 5**

**1**. Two-step binomial tree. In this case $u = 1.1$, $d = 0.9$, $r = 0.12$, $E = 42$. There are two time steps of $\Delta t = \frac{1}{4}$

The value of the risk-neutral probability.

$$p = \frac{\exp(0.12 \times 0.25) - 0.9}{1.1 - 0.9} = 0.6523.$$

The possible final asset prices are $40 \times (1.1)^2 = 48.4$, $40 \times 1.1 \times 0.9 = 39.6$ and $40 \times (0.9)^2 = 32.4$. The corresponding payoff: $p_{uu} = 0$, $p_{ud} = 42 - 39.6 = 2.4$, $p_{dd} = 42 - 32.4 = 9.6$.

Then for the European put option

$$p_u = \exp(-0.12 \times 0.25) \times [0.6523 \times 0 + (1 - 0.6523) \times 2.4] = 0.8098$$

$$p_d = \exp(-0.12 \times 0.25) \times [0.6523 \times 2.4 + (1 - 0.6523) \times 9.6] = 4.7585$$

Finally, the value of a 6-month European put option is

$$p_0 = \exp(-0.12 \times 0.25) \times [0.6523 \times 0.8098 + (1 - 0.6523) \times 4.7585] = 2.118\,3$$

For the American put option:

the payoff from early exercise at time $\Delta t$ is $42 - 40 \times 1.1 = -2.0$ and $42 - 40 \times 0.9 = 6.0$. In the later case early exercise is optimal $(6.0 > 4.7585)$, the value of the American put option should be 6 rather than 4.7585.

Therefore the value at $t = 0$ is

$$p_0 = \exp(-0.12 \times 0.25) \times [0.6523 \times 0.8098 + (1 - 0.6523) \times 6] = 2.5372$$

The payoff at $t = 0$ is $42 - 40 = 2$. It is not optimal.

**2**. Initial stock price is $S_0$. Stock price can either move up to $S_0u$ or down to $S_0d$. Consider a portfolio consisting of a long position in $\Delta$ shares and a short position in $N$ numbers of bonds: $\Pi = \Delta S - NB$. At maturity

$$\Pi(T) = \begin{cases} \Delta S_0 u - NB_0 e^{rT}, \\ \Delta S_0 d - NB_0 e^{rT}, \end{cases}.$$

Let us find $\Delta$ and $NB_0$ when

$$\Delta S_0 u - NB_0 e^{rT} = C_u,$$

$$\Delta S_0 d - NB_0 e^{rT} = C_d.$$

One can find that

$$\Delta = \frac{C_u - C_d}{S_0 u - S_0 d}$$

and

$$NB_0 = \frac{C_u d - C_d u}{u - d} e^{-rT}.$$

The cost of setting up this portfolio is $S_0\Delta - NB_0$. Therefore

$$C_0 = S_0\Delta - NB_0 = \frac{C_u - C_d}{u-d} - \frac{C_u d - C_d u}{u-d} e^{-rT}$$

or

$$C_0 = e^{-rT}\mathbf{E}\left[C_T\right] = e^{-rT}\left[pC_u + (1-p)C_d\right],$$

where

$$p = \frac{e^{rT} - d}{u-d}.$$

**3**. Since the put-call parity is violated, $7.78 < 5.09 - 20.37 + 24e^{-0.0748\times 0.5} = 7.8390$, there exist an arbitrage opportunity.

We set up the portfolio $\Pi = S + P - C - B$ such that $\Pi_0 = 20.37 + 7.78 - 5.09 - 23.06 = 0$. The balance is zero!

After six months:

$$\Pi_T = E - 23.06 \times e^{0.0748\times 0.5} = 24 - 23.06 \times e^{0.0748\times 0.5} \approx 0.06.$$

This is an arbitrage profit. Note that $S + P - C = E$ at $t = T$.