Introduction to Financial Mathematics - 20912

Lecturer - Prof. Sergei Fedotov

Exercise Sheet 4: Solutions

1. In this case $S_0 = 100$, $u = 1.1$, $d = 0.9$, $r = 0.05$, $T = 0.25$, $C_u = 15$, $C_d = 0$.

No-arbitrage arguments: the number of shares

$$\Delta = \frac{C_u - C_d}{S_0 u - S_0 d} = \frac{15 - 0}{100 \times 1.1 - 100 \times 0.9} = 0.75$$

and the value of call option

$$C_0 = S_0 \Delta - (S_0 u \Delta - C_u)\, e^{-rT} = 100 \times 0.75 - (100 \times 1.1 \times 0.75 - 15) \times e^{-0.05 \times 0.25} = 8.3385$$

Risk-neutral valuation: one can find the probability p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 0.25} - 0.9}{1.1 - 0.9} = 0.56289$$

and the value of call option

$$C_0 = e^{-rT}\left[pC_u + (1-p)C_d\right] = e^{-0.05 \times 0.25}\left[0.56289 \times 15 + 0\right] = 8.3385$$

2. In this case $u = \frac{85}{80}$, $d = \frac{75}{80}$, $r = 0.05$, $T = \frac{1}{3}$, $P_u = 0$, $P_d = 5$.

Risk-neutral valuation: one can find the probability p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times \frac{1}{3}} - \frac{75}{80}}{\frac{85}{80} - \frac{75}{80}} = 0.63445$$

and the value of put option

$$P_0 = e^{-rT}\left[pP_u + (1-p)P_d\right] = e^{-0.05 \times \frac{1}{3}}\left[0 + (1 - 0.63445) \times 5\right] = 1.7975$$

3. In this case $u = 1.1$, $d = 0.9$, $r = 0.12$, $\Delta t = 0.25$, $T = 2\Delta t$.

Risk-neutral valuation: one can find the probability p

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.65227.$$

The value of call option C_0 is

$$C_0 = e^{-r\Delta t}\left[pC_u + (1-p)C_d\right] = e^{-0.12 \times 0.25}\left[0.6523 \times 2.0256 + 0\right] = 1.2822$$

where

$$C_u = e^{-r\Delta t}\left[pC_{uu} + (1-p)C_{ud}\right] = e^{-0.12 \times 0.25}\left[0.65 \times 3.2 + 0\right] = 2.0256$$

$$C_d = e^{-r\Delta t}\left[pC_{ud} + (1-p)C_{dd}\right] = 0$$

(see two-step tree).

4. $\Delta t = 0.25$

The risk-neutral probability p is $\frac{e^{0.05\times0.25}-0.9}{1.2-0.9} = 0.375\,3$. So with probability $p^2 = 0.141$, $S_0u^2 = 40\times(1.2)^2 = 57.\,6$, with probability $2p(1-p) = 2\times0.375\,3\times(1-0.375\,3) = 0.4689$, $S_0ud = 40\times1.2\times0.9 = 43.\,2$ and with probability $(1-p)^2 = (1-0.375\,3)^2 = 0.390\,$,

$S_0d^2 = 40\times(0.9)^2 = 32.\,4$

5. We have three equations

$$qu + (1-q)d = e^{\mu\Delta t}, \qquad qu^2 + (1-q)d^2 - (qu + (1-q)d)^2 = \sigma^2\Delta t, \qquad d = u^{-1}$$

for three unknown parameters u, d and q. From the first equation we find

$$q = \frac{e^{\mu\Delta t} - d}{u - d}$$

Substituting from the last equation into the second equation, we get

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2\Delta t.$$

By using series expansions $e^{\mu\Delta t} \approx 1 + \mu\Delta t$, $e^{2\mu\Delta t} \approx 1 + 2\mu\Delta t$ and $d = u^{-1}$, we get an equation for u

$$(1 + \mu\Delta t)\left(u + u^{-1}\right) - 2 - 2\mu\Delta t = \sigma^2\Delta t$$

or

$$u^2 + 1 = u\frac{2 + 2\mu\Delta t + \sigma^2\Delta t}{1 + \mu\Delta t} = 2u + u\frac{\sigma^2\Delta t}{1 + \mu\Delta t} = u\left(2 + \sigma^2\Delta t\right) + o(\Delta t).$$

The purpose now is to find u as a function of $\sqrt{\Delta t}$. Last equation can be rewritten as quadratic equation: $u^2 - 2ku + 1 = 0$ with $k = 1 + \frac{\sigma^2\Delta t}{2}$. The solution is $u = k + \sqrt{(k^2-1)} > 1$. Substituting $k = 1 + \frac{\sigma^2\Delta t}{2}$ into $u = k + \sqrt{(k^2-1)}$, we get

$$u = 1 + \frac{\sigma^2\Delta t}{2} + \sqrt{\left(1 + \frac{\sigma^2\Delta t}{2}\right)^2 - 1} = 1 + \frac{\sigma^2\Delta t}{2} + \sqrt{\sigma^2\Delta t + \left(\frac{\sigma^2\Delta t}{2}\right)^2}.$$

If we expand this in a Taylor series in terms of $\sqrt{\Delta t}$, we obtain

$$u \approx 1 + \sigma\sqrt{\Delta t}.$$

Since $\sqrt{\Delta t}$ is small, we can write

$$u \approx e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t} \qquad \mathit{Cox, Ross, Rubinstein}(1979)$$