Introduction to Financial Mathematics - 20912

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Exercise Sheet 3: Solutions

1. (a) Consider $\Pi = S - C$. At maturity $t = T$

$$\Pi_T = S - \max(S - E, 0) = \begin{cases} S, \ S \le E, \\ E, \ S > E, \end{cases}$$

so $\Pi_T \ge 0$. Since a portfolio with positive terminal value must have positive value (by no-arbitrage) at any time: $\Pi_t = S_t - C_t \ge 0$, therefore $C_t \le S_t$.

(b) Consider the portfolio $\Pi = C - S + B$, where B is the risk-free bond with face value E at time T . It follows from

$$\Pi_T = \max(S - E, 0) - S + E = \begin{cases} -S + E, \ S \le E, \\ 0, \ S > E, \end{cases},$$

that $\Pi_T \ge 0$, and so (by no-arbitrage) at any time: $\Pi_t = C_t - S_t + E\exp(-r(T-t)) \ge 0$, i.e. $C_t \ge S_t - E\exp(-r(T-t))$.

(c) Consider the portfolio $\Pi = C(E_1) - C(E_2)$ with $E_1 < E_2$). At maturity $t = T$

$$\Pi_T = \begin{cases} 0, \ S_T \le E_1, \\ S - E_1, \ E_1 \le S < E_2, \\ E_2 - E_1, \ E_2 \le S. \end{cases}$$

It follows from this payoff that $0 \le \Pi_T \le E_2 - E_1$, hence $0 \le C_T(E_1) - C_T(E_2) \le E_2 - E_1$, and so at any time $t \le T$,

$$0 \le C_t(E_1) - C_t(E_2) \le (E_2 - E_1)\exp(-r(T-t)) \le E_2 - E_1.$$

2. (a) In this case $S_0 = 21$, $E = 15$, $T = 0.5$, and $r = 0.08$.

The lower bound for the call option price is $S_0 - E\exp(-rT)$, or $21 - 15\exp(-0.08 \times 0.5) = 6.588$.

(b) Investor could establish a zero initial investment $\Pi_0 = 0$ by purchasing one call for £5 (underpriced asset) and the bond for £16 and selling one share for £21.

At maturity $t = T$, the portfolio $\Pi = C + B - S$ has the value:

$$\Pi_T = \max(S - E, 0) + 16\exp(0.08 \times 0.5) - S = \begin{cases} 16.653 - S, \ S \le 15, \\ 1.653, \quad S > 15. \end{cases}$$

It is clear that $\Pi_T > 0$, therefore there exists an arbitrage opportunity.

3. By using Put-Call parity $P_0 = C_0 + E\exp(-rT) - S_0$, we find $P_0 \ge E\exp(-rT) - S_0$.

4. Assume that $P_0 > E\exp(-rT)$. Since the put option is overpriced, we sell it and invest the proceeds in risk-free bond. We set up a portfolio $\Pi = -P + B$. Of course $\Pi_0 = 0$ or $P_0 = B_0$ (no initial investments). At maturity

$$\Pi_T = -\max(E - S, 0) + P_0\exp(rT) = \begin{cases} S - E + P_0\exp(rT), \ S \le E, \\ P_0\exp(rT), \quad S > E. \end{cases}$$

Since $P_0\exp(rT) > E$, we conclude that $S - E + P_0\exp(rT) > 0$ for $S \le E$ and $\Pi_T > 0$. Therefore there exists an arbitrage opportunity.