Introduction to Financial Mathematics - 20912

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Exercise Sheet 1: Solutions

1. Equation for the interest rate r is $e^{r12} = 2$. Solution: $r \approx 0.05\,78$, that is, about 5.78%

2. The difference is $140e^{0.1} - 140\left(1 + \frac{0.1}{12}\right)^{12} \approx 0.06\,4$

3. In this case $S_0 = 90$, $\mu = 0.14$ and $\sigma = 0.20$. The time interval $\Delta t = 3/365$.

$$\Delta S = 90\left(0.14 \times \frac{3}{365} + 0.20\sqrt{\frac{3}{365}}X\right) \approx 0.104\ + 1.632X.$$

4. In this case $S_0 = 80$, $\mu = 0.36$ and $\sigma = 0.40$. The time interval $\Delta t = 1/365 = 2.\,74 \times 10^{-3}$.

$$S = 80 + 80\left(0.36 \times 2.\,74 \times 10^{-3} + 0.40\sqrt{2.\,74 \times 10^{-3}}X\right) \approx 80.\,079 + 1.\,675X.$$

5. By using Itó's Lemma

$$df = \left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dW,$$

we find

$$(a)\quad df = (2t + \mu AS)\,dt + \sigma ASdW = \left(2t + \mu\left(f - t^2\right)\right)dt + \sigma\left(f - t^2\right)dW,$$

$$(b)\ \ df = \left(\frac{1}{2}\mu - \frac{1}{8}\sigma^2\right)S^{1/2}dt + \sigma\frac{1}{2}S^{1/2}dW = \left(\frac{1}{2}\mu - \frac{1}{8}\sigma^2\right)fdt + \sigma\frac{1}{2}fdW;$$

$$(c)\ \ df = \left(\mu S(nS^{n-1}) + \frac{1}{2}\sigma^2 S^2(n(n-1)S^{n-2}\right)dt + \sigma S(nS^{n-1})dW = \left(\mu n + \frac{1}{2}\sigma^2 n(n-1)\right)fdt + \sigma nfdW.$$

6. We have $\ln S(t) = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W\left(t\right)$, therefore

$$\ln S \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right),$$

since $E\left\{W\left(t\right)\right\} = 0$ and $var\left\{\ln S(t)\right\} = E\left\{\sigma^2 W^2\left(t\right)\right\} = \sigma^2 t.$

In our case $S_0 = 90$, $\mu = 0.24$, $\sigma = 0.35$ and $T = \frac{2}{3}$:

$$\ln S \sim N\left(\ln 90 + \left(0.24 - (0.35)^2/2\right) \times \frac{2}{3}, (0.35)^2\,\frac{2}{3}\right)$$

or

$$\ln S \sim N\left(4.619, 0.082\right).$$

7. Let us find the first derivative $\frac{\partial p}{\partial t}$ by using product rule:

$$\frac{\partial p}{\partial t} = -\frac{1}{2\sqrt{2\pi}t^{\frac{3}{2}}}\exp\left(-\frac{y^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)\frac{y^2}{2t^2}$$

, $\frac{\partial p}{\partial y} = -\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)\frac{y}{t}$, and $\frac{\partial^2 p}{\partial y^2} = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)\frac{y^2}{t^2} - \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right)\frac{1}{t}$. Therefore

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\ .$$