Lecture 9

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20912 - Introduction to Financial Mathematics

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1. Risk-Neutral Valuation
2. Risk-Neutral World
3. Two-Steps Binomial Tree

Risk-Neutral Valuation

Reminder from Lecture 8. Call option price:

$$C_0 = e^{-rT}\left(pC_u + (1-p)C_d\right),$$

where $p = \frac{e^{rT}-d}{u-d}$. No-Arbitrage Principle: $d < e^{rT} < u$.

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The probability of up q or down movement $1 - q$ in the stock price plays no role whatsoever! Why???

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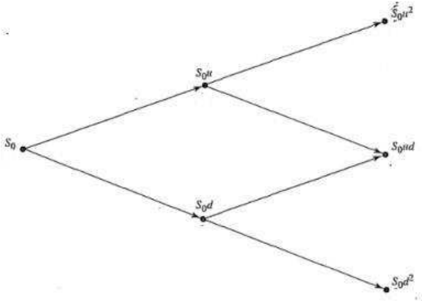
Risk-Neutral Valuation: $C_0 = e^{-rT}\mathbb{E}_p[C_T]$

The option price is the expected payoff in a risk-neutral world, discounted at risk-free rate r.

Two-Step Binomial Tree

Now the stock price changes twice, each time by either a factor of $u > 1$ or $d < 1$. We assume that the length of the time step is Δt such that $T = 2\Delta t$. After two time steps the stock price will be $S_0 u^2$, $S_0 ud$ or $S_0 d^2$.



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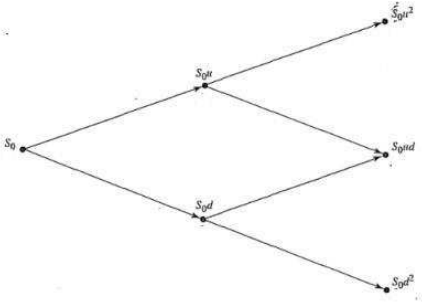

The call option expires after two time steps producing payoffs C_{uu}, C_{ud} and C_{dd} respectively.

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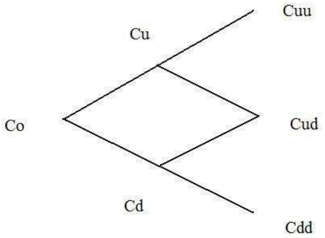

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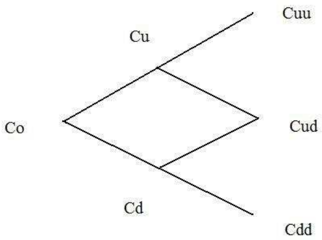

Current option price: $\quad C_0 = e^{-r\Delta t}\left(pC_u + (1-p)C_d\right).$

Option Price

Substitution gives

$$C_0 = e^{-2r\Delta t}\left(p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}\right),$$

where p^2, $2p(1-p)$ and $(1-p)^2$ are the probabilities in a risk-neutral world that the upper, middle, and lower final nodes are reached.

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The current put option price can be found in the same way:

$$P_0 = e^{-2r\Delta t}\left(p^2 P_{uu} + 2p(1-p)P_{ud} + (1-p)^2 P_{dd}\right)$$

or

$$P_0 = e^{-rT}\mathbb{E}_p\left[P_T\right].$$

Two-Step Binomial Tree Example

Consider six months European put with a strike price of £32 on a stock with current price £40. There are two time steps and in each time step the stock price either moves up by 20% or moves down by 20%. Risk-free interest rate is 10%. Find the current option price.

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Thus the value of put option is
$$P_0 = e^{-2\times 0.1\times 0.25} \times \left(0 + 0 + (1 - 0.5633)^2 \times 6.4\right) = 1.1610.$$