Lecture 8

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20912 - Introduction to Financial Mathematics

Lecture 8

1. One-Step Binomial Model for Option Price
2. Risk-Neutral Valuation
3. Examples

One-Step Binomial Model

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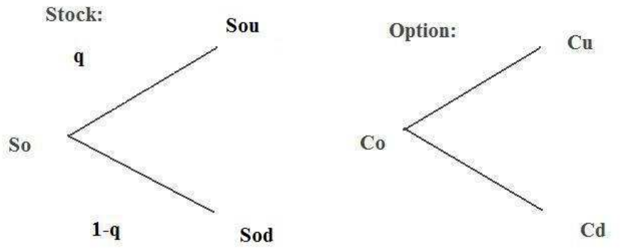

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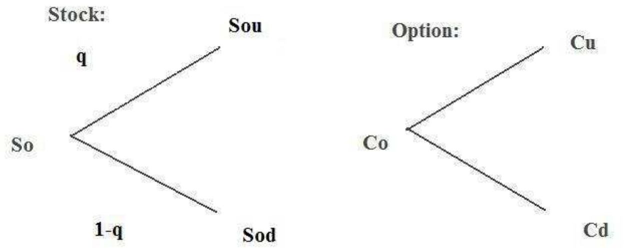

The purpose is to find the current price C_0 of a European call option.

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If portfolio $\Pi = \Delta S - C$ is risk-free, then $\Delta S_0 u - C_u = \Delta S_0 d - C_d$

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Finally, the current call option price is

$$C_0 = \Delta S_0 - (\Delta S_0 u - C_u)\, e^{-rT},$$

where $\Delta = \frac{C_u - C_d}{S_0(u-d)}$ (No-Arbitrage Argument).

Risk-Neutral Valuation

Alternatively

$$C_0 = e^{-rT}\left(pC_u + (1-p)C_d\right),$$

where

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Fair price of a call option C_0 is equal to the expected value of its future payoff discounted at the risk-free interest rate. For a put option P_0 we have the same result

$$P_0 = e^{-rT}\left(pP_u + (1-p)P_d\right).$$

Example

A stock price is currently \$40. At the end of three months it will be either \$44 or \$36. The risk-free interest rate is 12%.
What is the value of three-month European call option with a strike price of \$42? Use *no-arbitrage* arguments and *risk-neutral valuation*.

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and the value of call option

$$C_0 = S_0 \Delta - (S_0 u \Delta - C_u)\, e^{-rT} =$$

$$40 \times 0.25 - (40 \times 1.1 \times 0.25 - 2) \times e^{-0.12 \times 0.25} = 1.266$$

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Risk-neutral valuation: one can find the probability p

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

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$$C_0 = e^{-rT}\left[pC_u + (1-p)C_d\right] = e^{-0.12\times 0.25}\left[0.6523 \times 2 + 0\right] = 1.266$$