Lecture 7

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20912 - Introduction to Financial Mathematics

Lecture 7

1. Upper and Lower Bounds on Put Options
2. Proof of Put-Call Parity by No-Arbitrage Principle
3. Example on Arbitrage Opportunity

Upper and Lower Bounds on Put Option

Reminder from lecture 6.

• Arbitrage opportunity arises when a zero initial investment $\Pi_0 = 0$ is identified that guarantees a non-negative payoff in the future such that $\Pi_T > 0$ with non-zero probability.

Upper and Lower Bounds on Put Option

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• Arbitrage opportunity arises when a zero initial investment $\Pi_0 = 0$ is identified that guarantees a non-negative payoff in the future such that $\Pi_T > 0$ with non-zero probability.

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Upper and Lower Bounds on Put Option (exercise sheet 3):

$$Ee^{-rT} - S_0 \leq P_0 \leq Ee^{-rT}$$

Let us illustrate these bounds geometrically.

Proof of Put-Call Parity

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We set up the portfolio $\Pi = -P - S + C + B$. At time $t = 0$ we

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The balance of all these transactions is zero, that is, $\Pi_0 = 0$

Proof of Put-Call Parity

At maturity $t = T$ the portfolio $\Pi = -P - S + C + B$ has the value

$$\Pi_T = \begin{cases} -(E - S) - S + B_0 e^{rT}, & S \leq E, \\ -S + (S - E) + B_0 e^{rT}, & S > E, \end{cases} = -E + B_0 e^{rT}$$

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Since $B_0 > Ee^{-rT}$, we conclude $\Pi_T > 0$. and $\Pi_0 = 0$.

This is an arbitrage opportunity.

Proof of Put-Call Parity

Now we assume that $P_0 < C_0 - S_0 + Ee^{-rT}$.

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- buy one share for S_0 (long position)

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At time $t = 0$ we

- buy one put option for P_0
- buy one share for S_0 (long position)
- sell one call option for C_0 (write the call option)
- borrow $B_0 = P_0 + S_0 - C_0 < Ee^{-rT}$

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The balance of all these transactions is zero, that is, $\Pi_0 = 0$

At maturity $t = T$ we have $\Pi_T = E - B_0 e^{rT}$. Since $B_0 < Ee^{-rT}$, we conclude $\Pi_T > 0$.

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Example on Arbitrage Opportunity

Three months European call and put options with the exercise price £12 are trading at £3 and £6 respectively.

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$6 < 3 - 8 + 12e^{-0.05 \times \frac{1}{4}} = 6.851$

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- buy a put option for £6
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To get arbitrage profit we
- buy a put option for £6
- sell a call option for £3
- buy a share for £8
- borrow £11 at the interest rate 5%.

The balance is zero!!

Example: Arbitrage Opportunity

The value of the portfolio $\Pi = P + S - C - B$ at maturity $T = \frac{1}{4}$ is $\Pi_T = E - B_0 e^{rT} = 12 - 11e^{0.05\times\frac{1}{4}} \approx 0.862.$

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Combination $P + S - C$ gives us £12. We repay the loan £$11e^{0.05\times\frac{1}{4}}$.

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Combination $P + S - C$ gives us £12. We repay the loan £$11e^{0.05\times\frac{1}{4}}$.

The balance $12 - 11e^{0.05\times\frac{1}{4}}$ is an arbitrage profit £0.862.