Lecture 6

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20912 - Introduction to Financial Mathematics

Lecture 6

1. No-Arbitrage Principle
2. Put-Call Parity
3. Upper and Lower Bounds on Call Options

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• Let Π_t be the value of the portfolio at time t. If $\Pi_T \geq 0$, then $\Pi_t \geq 0$ for $t < T$.

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Using No Arbitrage Principle, we obtain

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This relationship between S_t, P_t and C_t is called Put-Call Parity which represents an example of complete risk elimination.

Upper and Lower Bounds on Call Options

- Put-Call Parity ($t = 0$): $S_0 + P_0 - C_0 = Ee^{-rT}$.

It shows that the value of European call option can be found from the value of European put option with the same strike price and maturity:

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Therefore $C_0 \geq S_0 - Ee^{-rT}$ since $P_0 \geq 0$ $S_0 - Ee^{-rT}$ is the lower bound

for call option.

$$S_0 - Ee^{-rT} \leq C_0 \leq S_0$$

Let us illustrate these bounds geometrically.

Examples

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We establish a zero initial investment $\Pi_0 = 0$ by purchasing one call for £4 and the bond for £36 and selling one share for £40. The portfolio is $\Pi = C + B - S$.

Examples

At maturity $t = T$, the portfolio $\Pi = C + B - S$ has the value:

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It is clear that $\Pi_T > 0$, therefore there exists an arbitrage opportunity.