Lecture 3

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20912 - Introduction to Financial Mathematics

Lecture 3

1. Distribution for $\ln S(t)$
2. Solution to Stochastic Differential Equation for Stock Price
3. Examples

Differential of $\ln S$

Example 1. Find the stochastic differential equation (SDE) for

$$f = \ln S$$

by using Itô's Lemma:

$$df = \left(\frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW$$

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Integration from 0 to t gives

$$f - f_0 = \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \qquad \textit{since} \qquad W(0) = 0.$$

Normal distribution for $\ln S(t)$

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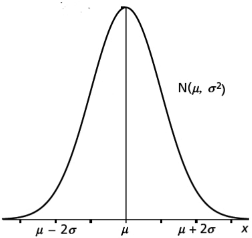
Answer: $\ln S(0.5) \sim N(3.759, 0.020)$

Probability density function for $\ln S(t)$

Recall that if the random variable X has a normal distribution with mean μ and variance σ^2, then the probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



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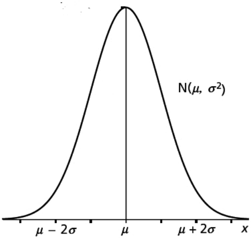

The probability density function of $X = \ln S(t)$ is

$$\frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x - \ln S_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t}\right)$$

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Or

$$S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma \sqrt{t} X} \qquad where \qquad X \sim N(0,1)$$