Lecture 2

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20912 - Introduction to Financial Mathematics

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1. Properties of Wiener Process
2. Approximation for Stock Price Equation
3. Itô's Lemma

Wiener process

The probability density function for $W(t)$ is

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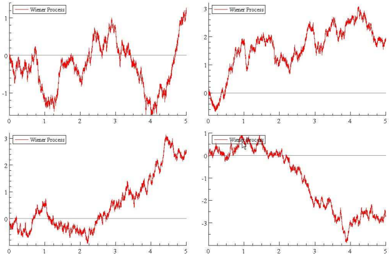
and $\mathbb{P}\left(a \leq W(t) \leq b\right) = \int_a^b p(y,t)dy$

- Simulations of a Wiener process:



Approximation of SDE for small Δt

• The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X\,(\Delta t)^{\frac{1}{2}}$, where X is a random variable with normal distribution with zero mean and unit variance: $X \sim N\,(0, 1)$

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Recall: equation for the stock price is

$$dS = \mu S dt + \sigma S dW,$$

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It means $\Delta S \sim N\left(\mu S \Delta t, \sigma^2 S^2 \Delta t\right)$

Examples

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Example 2. Show that the return $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$

Itô's Lemma

We assume that $f(S,t)$ is a smooth function of S and t.

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Example. Find the SDE satisfied by $f = S^2$.