Lecture 19

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20912 - Introduction to Financial Mathematics

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1. Asian Options

2. Derivation of PDE for Option Price

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• Final condition can be rewritten as

$$V(S,I,T) = \max\left(S - \frac{I}{T}, 0\right).$$

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The change in the value of this portfolio in the time interval dt:

$$d\Pi = dV - \Delta dS.$$

Itô's Lemma and Elimination of Risk

Using Itô's Lemma:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial I} dI$$

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We can eliminate the random component in $d\Pi$ by choosing $\Delta = \frac{\partial V}{\partial S}$.

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The value of Asian option must be calculated numerically (no analytical solution!)