Lecture 18

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20912 - Introduction to Financial Mathematics

Lecture 18

1. Measure of Future Values of Interest Rate
2. Term Structure of Interest Rate (Yield Curve)

Measure of Future Value of Interest Rate

Let us consider the case when the dividend payment $K(t) = 0$. Recall that the solution of the equation $\frac{dV}{dt} = r(t)V$ with $V(T) = F$ is

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since $\frac{\partial}{\partial T}\int_t^T r(s)ds = r(T)$. Therefore,

$$r(T) = -\frac{1}{V(t, T)}\frac{\partial V}{\partial T}.$$

This is an interest rate at future dates (forward rate).

Term Structure of Interest Rate (Yield Curve)

We define

$$Y(t, T) = -\frac{\ln(V(t, T)) - \ln V(T, T)}{T - t}$$

as a measure of the future values of interest rate, where $V(t, T)$ is taken from financial data.

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Term structure of interest rate (yield curve):

$$Y(0,T) = -\frac{\ln(V(0,T)) - \ln V(T,T)}{T} = \frac{1}{T}\int_0^T r(s)ds$$

is the average value of interest rate in the future.

Example

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Let us introduce the probability of default p during one year $T = 1$. Then

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