Lecture 17

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20912 - Introduction to Financial Mathematics

Lecture 17

1. Bond Pricing with Known Interest Rates and Dividend Payments
2. Zero-Coupon Bond Pricing

Bond Pricing Equation

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Equation for the bond price is

$$\frac{dV}{dt} = r(t)V - K(t).$$

The final condition: $V(T) = F$.

Zero-Coupon Bond Pricing

Let us consider the case when the dividend payment $K(t) = 0$.

The solution of the equation $\frac{dV}{dt} = r(t)V$ with $V(T) = F$ can be written as

$$V(t) = F \exp\left(-\int_t^T r(s)ds\right).$$

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$\ln V(T) - \ln V(t) = \int_t^T r(s)ds$ or $\ln(V(t)/F) = -\int_t^T r(s)ds$.

Example

A zero coupon bond, V, issued at $t = 0$, is worth $V(1) = 1$ at $t = 1$. Find the bond price $V(t)$ at time $t < 1$ and $V(0)$, when the continuous interest rate is

$$r(t) = t^2.$$

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Therefore

$$\begin{aligned} V(t) &= \exp\left(-\int_t^1 r(s)ds\right) = \exp\left(\frac{t^3}{3} - \frac{1}{3}\right), \\ V(0) &= \exp\left(-\frac{1}{3}\right) = 0.7165 \end{aligned}$$

Bond Pricing for $K(t) > 0$

Let us consider the case when the dividend payment $K(t) > 0$. The solution of the equation $\frac{dV}{dt} = r(t)V - K(t)$ can be written as

$$V(t) = F \exp\left(-\int_t^T r(s)ds\right) + V_1(t),$$

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where $V_1(t) = C(t) \exp\left(-\int_t^T r(s)ds\right)$.

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To find $C(t)$ we need to substitute $V_1(t)$ into the equation

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(tutorial exercise 8)

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The explicit solution is

$$V(t) = \exp\left(-\int_t^T r(s)ds\right)\left[F + \int_t^T K(y)\exp\left(\int_y^T r(s)ds\right)dy\right].$$