Lecture 16

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20912 - Introduction to Financial Mathematics

Lecture 16

1. Option on Dividend-paying Stock
2. American Put Option

Option on Dividend-paying Stock

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The change in the value of this portfolio in the time interval dt:

$$d\Pi = dC - \Delta dS - \Delta D_0 S dt.$$

Itô's Lemma and Elimination of Risk

Using Itô's Lemma:

$$dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS$$

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we find

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S} \right) dt + \frac{\partial C}{\partial S} dS - \Delta dS$$

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We can eliminate the random component in $d\Pi$ by choosing $\Delta = \frac{\partial C}{\partial S}$.

Modified Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = C - S\frac{\partial C}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S}\right) dt$.

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Thus, we obtain the modified Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC = 0.$$

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If we substitute $C(S,t) = e^{-D_0(T-t)} C_1(S,t)$ into the modified Black-Scholes equation, we find the equation for $C_1(S,t)$ in the form

$$\frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r - D_0)\frac{\partial C_1}{\partial S} - (r - D_0)C_1 = 0$$

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. The auxiliary function $C_1(S,t)$ is the value of a European Call option with the interest rate $r - D_0$.

Solution to Modified Black-Scholes Equation

Problem sheet 7: show that the modified Black-Scholes equation has the explicit solution for the European call

$$C(S,t) = Se^{-D_0(T-t)}N(d_{10}) - Ee^{-r(T-t)}N(d_{20}),$$

where

$$d_{10} = \frac{\ln(S/E) + \left(r - D_0 + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_{20} = d_{10} - \sigma\sqrt{T-t}.$$

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• The American put option value must be greater than or equal to the payoff function.

If $P < \max(E - S, 0)$, then there is obvious arbitrage opportunity.

American Put Option

American put problem can be written as as a free boundary problem.

We divide the price axis S into two distinct regions:

$$0 \leq S < S_f(t) \ \ and \ \ S_f(t) < S < \infty,$$

where $S_f(t)$ is the exercise boundary. Note that we do not know a priori the value of $S_f(t)$.

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$$P\left(S_f(t), t\right) = \max\left(E - S_f(t), 0\right), \ \ \frac{\partial P}{\partial S}\left(S_f(t), t\right) = -1.$$