Lecture 15

Sergei Fedotov

20912 - Introduction to Financial Mathematics

Lecture 15

1. Black-Scholes Equation and Replicating Portfolio

2. Static and Dynamic Risk-Free Portfolio

Replicating Portfolio

The aim is to show that the option price $V(S, t)$ satisfies the Black-Scholes equation

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How to find (Δ, N) such that $\Pi_t = V_t$?

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- SDE for a stock price $S(t)$: $\quad dS=\mu S dt+\sigma S dW.$

- Equation for a bond price $B(t)$: $\quad dB=rBdt.$

Derivation of the Black-Scholes Equation

By using the Ito's lemma, we find the change in the option value

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW.$$

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Since $NB = \Delta S - \Pi = \left(S \frac{\partial V}{\partial S} - V \right)$, we get the classical Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

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Definition: The risk of a portfolio is the variance of the return.

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We can eliminate the random component in Π by choosing

$$\Delta = \frac{\partial C}{\partial S}.$$

This is a Δ-hedging! It requires a continuous rebalancing of a number of shares in the portfolio Π.