Lecture 13

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20912 - Introduction to Financial Mathematics

1. Boundary Conditions for Call and Put Options

2. Exact Solution to Black-Scholes Equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

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$P(S,t) \to 0$ as $S \to \infty$.
As stock price $S \to \infty$, then put option is unlikely to be exercised.

Exact solution

The Black-Scholes equation

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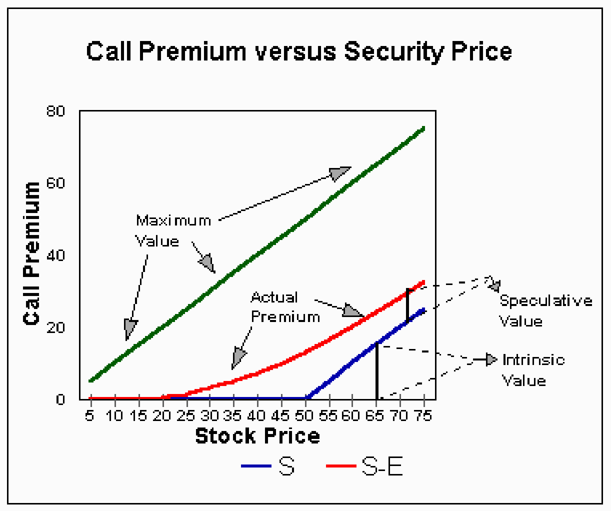
and

$$d_1=\frac{\ln\left(S/E\right)+\left(r+\sigma^2/2\right)\left(T-t\right)}{\sigma\sqrt{T-t}},\quad d_2=d_1-\sigma\sqrt{T-t}.$$

The European call value $C(S, t)$ by K. Rubash



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$$d_1 = \frac{\ln(S_0/E) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{21.6}{25}\right) + (0.01 + (0.35)^2 \times 0.5)) \times 0.25}{0.35 \times \sqrt{0.25}} \approx -0.7335$$

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Since

$$N(-0.7335) \approx 0.2316, \quad N(-0.9085) \approx 0.1818,$$

we obtain

$$C_0 \approx 21.6 \times 0.2316 - 25 \times e^{-0.01 \times 0.25} \times 0.1818 = 0.4689$$

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Therefore $\lim_{\sigma \to \infty} C(S,t) = S$. This is an upper bound for call option!!!