# Lecture 12

Sergei Fedotov

20912 - Introduction to Financial Mathematics

1. Black-Scholes Model
2. Black-Scholes Equation

The Black-Scholes model for option pricing was developed by Fischer Black, Myron Scholes in the early 1970s. This model is the most important result in financial mathematics.

# Black - Scholes model

The Black-Scholes model is used to calculate an option price using: stock price $S$, strike price $E$, volatility $\sigma$, time to expiration $T$, and risk-free interest rate $r$.

This model involves the following explicit assumptions:

• The stock price follows a Geometric Brownian motion with constant expected return and volatility: $dS = \mu S dt + \sigma S dW$. .

• No transaction costs.

# Black - Scholes model

The Black-Scholes model is used to calculate an option price using: stock price $S$, strike price $E$, volatility $\sigma$, time to expiration $T$, and risk-free interest rate $r$.

This model involves the following explicit assumptions:

• The stock price follows a Geometric Brownian motion with constant expected return and volatility: $dS = \mu S dt + \sigma S dW$. .

• No transaction costs.

• The stock does not pay dividends.

• One can borrow and lend cash at a constant risk-free interest rate.

# Black - Scholes model

The Black-Scholes model is used to calculate an option price using: stock price $S$, strike price $E$, volatility $\sigma$, time to expiration $T$, and risk-free interest rate $r$.

This model involves the following explicit assumptions:

• The stock price follows a Geometric Brownian motion with constant expected return and volatility: $dS = \mu S dt + \sigma S dW$. .

• No transaction costs.

• The stock does not pay dividends.

• One can borrow and lend cash at a constant risk-free interest rate.

• Securities are perfectly divisible (i.e. one can buy any fraction of a share of stock).

# Black - Scholes model

The Black-Scholes model is used to calculate an option price using: stock price $S$, strike price $E$, volatility $\sigma$, time to expiration $T$, and risk-free interest rate $r$.

This model involves the following explicit assumptions:

• The stock price follows a Geometric Brownian motion with constant expected return and volatility: $dS = \mu S dt + \sigma S dW$. .

• No transaction costs.

• The stock does not pay dividends.

• One can borrow and lend cash at a constant risk-free interest rate.

• Securities are perfectly divisible (i.e. one can buy any fraction of a share of stock).

• No restrictions on short selling.

## Basic Notation

We denote by $V(S, t)$ the value of an option. We use the notations $C(S, t)$ and $P(S, t)$ for call and put when the distinction is important.

# Basic Notation

We denote by $V(S,t)$ the value of an option. We use the notations $C(S,t)$ and $P(S,t)$ for call and put when the distinction is important.

- The aim is to derive the famous Black-Scholes Equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

# Basic Notation

We denote by $V(S,t)$ the value of an option. We use the notations $C(S,t)$ and $P(S,t)$ for call and put when the distinction is important.

- The aim is to derive the famous Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Now, we set up a portfolio consisting of a long position in one option and a short position in $\Delta$ shares.

The value is $\Pi = V - \Delta S$.

# Basic Notation

We denote by $V(S,t)$ the value of an option. We use the notations $C(S,t)$ and $P(S,t)$ for call and put when the distinction is important.

• The aim is to derive the famous Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Now, we set up a portfolio consisting of a long position in one option and a short position in $\Delta$ shares.

The value is $\Pi = V - \Delta S$.

• Let us find the number of shares $\Delta$ that makes this portfolio riskless.

# Itô's Lemma and Elimination of Risk

The change in the value of this portfolio in the time interval $dt$:
$d\Pi = dV - \Delta dS, \quad$ where $dS = \mu S dt + \sigma S dW$.

# Itô's Lemma and Elimination of Risk

The change in the value of this portfolio in the time interval $dt$:
$d\Pi = dV - \Delta dS$, where $dS = \mu S dt + \sigma S dW$.

Using Itô's Lemma:

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW,$$

# Itô's Lemma and Elimination of Risk

The change in the value of this portfolio in the time interval $dt$:
$d\Pi = dV - \Delta dS, \quad$ where $dS = \mu S dt + \sigma S dW$.

Using Itô's Lemma:

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW,$$

we find

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \mu S \Delta \right) dt + \left( \sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dW.$$

The main question is how to eliminate the risk!!!

# Itô's Lemma and Elimination of Risk

The change in the value of this portfolio in the time interval $dt$:
$d\Pi = dV - \Delta dS, \quad$ where $dS = \mu S dt + \sigma S dW$.

Using Itô's Lemma:

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW,$$

we find

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \mu S \Delta \right) dt + \left( \sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dW.$$

The main question is how to eliminate the risk!!!

We can eliminate the random component in $d\Pi$ by choosing $\Delta = \frac{\partial V}{\partial S}$.

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

- No-Arbitrage Principle: the return from this portfolio must be $rdt$.

$\frac{d\Pi}{\Pi} = rdt$

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be $rdt$.

$\frac{d\Pi}{\Pi} = rdt$ or $\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r$

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be $rdt$.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r \quad \text{or} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right).$$

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be $rdt$.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r \quad \text{or} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right).$$

Thus, we obtain the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be $rdt$.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r \quad \text{or} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right).$$

Thus, we obtain the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Scholes received the 1997 Nobel Prize in Economics.

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be *rdt*.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r \quad \text{or} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right).$$

Thus, we obtain the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Scholes received the 1997 Nobel Prize in Economics. It was not awarded to Black in 1997, because he died in 1995.

# Black-Scholes Equation

This choice results in a risk-free portfolio $\Pi = V - S\frac{\partial V}{\partial S}$ whose increment is $d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt$.

• No-Arbitrage Principle: the return from this portfolio must be *rdt*.

$$\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - S\frac{\partial V}{\partial S}} = r \quad \text{or} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right).$$

Thus, we obtain the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Scholes received the 1997 Nobel Prize in Economics. It was not awarded to Black in 1997, because he died in 1995.
Black received a Ph.D. in applied mathematics from Harvard University.

# The European call value $C(S, t)$

If PDE is of backward type, we must impose a final condition at $t = T$.

# The European call value $C(S, t)$

If PDE is of backward type, we must impose a final condition at $t = T$. For a call option, we have $C(S, T) = \max(S - E, 0)$.