Lecture 11

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20912 - Introduction to Financial Mathematics

1. American Put Option Pricing on Binomial Tree
2. Replicating Portfolio

American Option

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$E - (P + S) > 0$

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Here $0 \le n \le m$ and the risk-neutral probability $p = \frac{e^{r\Delta t}-d}{u-d}$.

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where S_n^m is the n-th possible value of stock price at time-step $m\Delta t$.

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where S_n^m is the n-th possible value of stock price at time-step $m\Delta t$.

- Final condition: $P_n^N = \max\left(E - S_n^N, 0\right)$, where $n = 0, 1, 2, ..., N$, E is the strike price.

Example: Evaluation of American Put Option on Two-Step Tree

We assume that over each of the next two years the stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.

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Find the value of a 2-year American put with a strike price of \$52 on a stock whose current price is \$50.

In this case $u = 1.2$, $d = 0.8$, $r = 0.05$, $E = 52$.

Risk-neutral probability: $p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$

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Payoff: $E - S = 52 - 50 = 2 < 5.0894$. Early exercise is not optimal at the initial node

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The aim is to calculate the value of call option C_0.

Let us establish a portfolio of stocks and bonds in such a way that the payoff of a call option is completely replicated.

Final value: $\Pi_T = C_T = \max(S - E, 0)$

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Consider replicating portfolio of Δ shares held long and N bonds held short.
The value of portfolio: $\Pi = \Delta S - NB$. A pair (Δ, N) is called a trading strategy.

How to find (Δ, N) such that $\Pi_T = C_T$ and $\Pi_0 = C_0$?

Example: One-Step Binomial Model.

Initial stock price is S_0. The stock price can either move up from S_0 to S_0u or down from S_0 to S_0d. At time T, let the option price be C_u if the stock price moves up, and C_d if the stock price moves down.

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Current value: $C_0 = \Delta S_0 - NB_0$.

Prove: $C_0 = e^{-rT}\left(pC_u + (1-p)C_d\right)$, where $p = \frac{e^{rT}-d}{u-d}$. (Exercise sheet 5)