Lecture 10

Sergei Fedotov

20912 - Introduction to Financial Mathematics

Lecture 10

1. Binomial Model for Stock Price
2. Option Pricing on Binomial Tree
3. Matching Volatility σ with u and d

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$
- The price either moves up $S \to Su$ or down $S \to Sd$ with $d < e^{r\Delta t} < u$.

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$
- The price either moves up $S \to Su$ or down $S \to Sd$ with $d < e^{r\Delta t} < u$.
- The probability of up movement is q.

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t,\, 2\Delta t,\, 3\Delta t, \ldots$
- The price either moves up $S \to Su$ or down $S \to Sd$ with $d < e^{r\Delta t} < u$.
- The probability of up movement is q.

Let us build up a tree of possible stock prices. The tree is called a binomial tree, because the stock price will either move up or down at the end of each time period. Each node represents a possible future stock price.

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$
- The price either moves up $S \to Su$ or down $S \to Sd$ with $d < e^{r\Delta t} < u$.
- The probability of up movement is q.

Let us build up a tree of possible stock prices. The tree is called a binomial tree, because the stock price will either move up or down at the end of each time period. Each node represents a possible future stock price.

We divide the time to expiration T into several time steps of duration $\Delta t = T/N$, where N is the number of time steps in the tree.

Binomial model for the stock price

Continuous random model for the stock price: $dS = \mu S dt + \sigma S dW$

The binomial model for the stock price is a discrete time model:

- The stock price S changes only at discrete times $\Delta t, 2\Delta t, 3\Delta t, ...$
- The price either moves up $S \to Su$ or down $S \to Sd$ with $d < e^{r\Delta t} < u$.
- The probability of up movement is q.

Let us build up a tree of possible stock prices. The tree is called a binomial tree, because the stock price will either move up or down at the end of each time period. Each node represents a possible future stock price.

We divide the time to expiration T into several time steps of duration $\Delta t = T/N$, where N is the number of time steps in the tree.

Example: Let us sketch the binomial tree for $N = 4$.

Stock Price Movement in the Binomial Model

We introduce the following notations:

- S_n^m is the n-th possible value of stock price at time-step $m\Delta t$.

Stock Price Movement in the Binomial Model

We introduce the following notations:

- S_n^m is the n-th possible value of stock price at time-step $m\Delta t$.

Then $S_n^m = u^n d^{m-n} S_0^0, \quad$ where $n = 0, 1, 2, ..., m.$

S_0^0 is the stock price at the time $t = 0$. Note that u and d are the same at every node in the tree.

We introduce the following notations:

- S_n^m is the n-th possible value of stock price at time-step $m\Delta t$.

Then $S_n^m = u^n d^{m-n} S_0^0, \quad$ where $n = 0, 1, 2, ..., m$.

S_0^0 is the stock price at the time $t = 0$. Note that u and d are the same at every node in the tree.

For example, at the third time-step $3\Delta t$, there are four possible stock prices: $S_0^3 = d^3 S_0^0$, $S_1^3 = ud^2 S_0^0$, $S_2^3 = u^2 d S_0^0$ and $S_3^3 = u^3 S_0^0$.

At the final time-step $N\Delta t$, there are $N + 1$ possible values of stock price.

Call Option Pricing on Binomial Tree

We denote by C_n^m the n-th possible value of call option at time-step $m\Delta t$.

Call Option Pricing on Binomial Tree

We denote by C_n^m the n-th possible value of call option at time-step $m\Delta t$.

- Risk Neutral Valuation (backward in time):

$$C_n^m = e^{-r\Delta t}\left(pC_{n+1}^{m+1} + (1-p)C_n^{m+1}\right).$$

Here $0 \leq n \leq m$ and $p = \frac{e^{r\Delta t}-d}{u-d}$.

Call Option Pricing on Binomial Tree

We denote by C_n^m the n-th possible value of call option at time-step $m\Delta t$.

• Risk Neutral Valuation (backward in time):

$$C_n^m = e^{-r\Delta t}\left(pC_{n+1}^{m+1} + (1-p)C_n^{m+1}\right).$$

Here $0 \leq n \leq m$ and $p = \frac{e^{r\Delta t} - d}{u-d}$.

• Final condition: $C_n^N = \max\left(S_n^N - E, 0\right)$, where $n = 0, 1, 2, ..., N,$ E is the strike price.

We denote by C_n^m the n-th possible value of call option at time-step $m\Delta t$.

• Risk Neutral Valuation (backward in time):

$$C_n^m = e^{-r\Delta t}\left(pC_{n+1}^{m+1} + (1-p)C_n^{m+1}\right).$$

Here $0 \le n \le m$ and $p = \frac{e^{r\Delta t}-d}{u-d}$.

• Final condition: $C_n^N = \max\left(S_n^N - E, 0\right)$, where $n = 0, 1, 2, ..., N$, E is the strike price.

The current option price C_0^0 is the expected payoff in a risk-neutral world, discounted at risk-free rate r: $C_0^0 = e^{-rT}\mathbb{E}_p\left[C_T\right].$

Example: $N = 4$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

- Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}[S] = qS_0 u + (1-q)S_0 d$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}[S] = qS_0 u + (1-q)S_0 d$.

First equation: $qu + (1-q)d = e^{\mu \Delta t}$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}[S] = qS_0 u + (1-q)S_0 d$.

First equation: $qu + (1-q)d = e^{\mu \Delta t}$.

• Variance of the return: Continuous model: $var\left[\frac{\Delta S}{S}\right] = \sigma^2 \Delta t$ (Lecture2)

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}\left[S\right]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}\left[S\right] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}\left[S\right] = qS_0 u + (1-q)S_0 d$.

First equation: $qu + (1-q)d = e^{\mu \Delta t}$.

• Variance of the return: Continuous model: $var\left[\frac{\Delta S}{S}\right] = \sigma^2 \Delta t$ (Lecture2)
On the binomial tree: $var\left[\frac{\Delta S}{S}\right] =$
$q(u-1)^2 + (1-q)(d-1)^2 - \left[q(u-1) + (1-q)(d-1)\right]^2 =$
$qu^2 + (1-q)d^2 - \left[qu + (1-q)d\right]^2$.

Recall: $var\left[X\right] = \mathbb{E}\left[X^2\right] - \left[\mathbb{E}\left(X\right)\right]^2$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}[S] = qS_0 u + (1-q)S_0 d$.

First equation: $qu + (1-q)d = e^{\mu \Delta t}$.

• Variance of the return: Continuous model: $var\left[\frac{\Delta S}{S}\right] = \sigma^2 \Delta t$ (Lecture2)
On the binomial tree: $var\left[\frac{\Delta S}{S}\right] =$
$q(u-1)^2 + (1-q)(d-1)^2 - [q(u-1) + (1-q)(d-1)]^2 =$
$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2$.

Recall: $var[X] = \mathbb{E}\left[X^2\right] - [\mathbb{E}(X)]^2$.

Second equation: $qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2 \Delta t$.

Matching volatility σ with u and d

We assume that the stock price starts at the value S_0 and the time step is Δt. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $var\left[\frac{\Delta S}{S}\right]$, for continuous and discrete models.

• Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$.
On the binomial tree: $\mathbb{E}[S] = qS_0 u + (1-q)S_0 d$.

First equation: $qu + (1-q)d = e^{\mu \Delta t}$.

• Variance of the return: Continuous model: $var\left[\frac{\Delta S}{S}\right] = \sigma^2 \Delta t$ (Lecture2)
On the binomial tree: $var\left[\frac{\Delta S}{S}\right] =$
$q(u-1)^2 + (1-q)(d-1)^2 - [q(u-1) + (1-q)(d-1)]^2 =$
$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2$.

Recall: $var[X] = \mathbb{E}\left[X^2\right] - [\mathbb{E}(X)]^2$.

Second equation: $qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2 \Delta t$.
Third equation: $u = d^{-1}$.

Matching volatility σ with u and d

From the first equation we find $q = \frac{e^{\mu \Delta t} - d}{u - d}$.

This is the probability of an up movement in the real world.

Matching volatility σ with u and d

From the first equation we find $q = \frac{e^{\mu\Delta t} - d}{u - d}$.

This is the probability of an up movement in the real world. Substituting this probability into the second equation, we obtain

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2 \Delta t.$$

Matching volatility σ with u and d

From the first equation we find $q = \frac{e^{\mu\Delta t} - d}{u - d}$.

This is the probability of an up movement in the real world. Substituting

this probability into the second equation, we obtain

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2 \Delta t.$$

Using $u = d^{-1}$, we get

$$e^{\mu\Delta t}(u + \frac{1}{u}) - 1 - e^{2\mu\Delta t} = \sigma^2 \Delta t.$$

This equation can be reduced to the quadratic equation. (Exercise sheet 4, part 5).

Matching volatility σ with u and d

From the first equation we find $q = \frac{e^{\mu\Delta t} - d}{u - d}$.

This is the probability of an up movement in the real world. Substituting

this probability into the second equation, we obtain

$$e^{\mu\Delta t}(u + d) - ud - e^{2\mu\Delta t} = \sigma^2 \Delta t.$$

Using $u = d^{-1}$, we get

$$e^{\mu\Delta t}(u + \frac{1}{u}) - 1 - e^{2\mu\Delta t} = \sigma^2 \Delta t.$$

This equation can be reduced to the quadratic equation. (Exercise sheet 4, part 5).

One can obtain $u \approx e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t}$ and $d \approx e^{-\sigma\sqrt{\Delta t}}$.

These are the values of u and d obtained by Cox, Ross, and Rubinstein in 1979.

Recall: $e^x \approx 1 + x$ for small x.