Lecture 1

Sergei Fedotov

20912 - Introduction to Financial Mathematics

No tutorials in the first week

Plan de la présentation

1. Introduction
   - Elementary economics background
   - What is financial mathematics?
   - The role of SDE's and PDE's

2. Time Value of Money

3. Continuous Model for Stock Price

General Information

Textbooks:

• J. Hull, Options, Futures and Other Derivatives, 7th Edition, Prentice-Hall, 2008.
• P. Wilmott, S. Howison and J. Dewynne, The Mathematics of Financial Derivatives: A Student Introduction, Cambridge University Press, 1995.

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Assessment:

In class test: 20% (12 March, Monday, week 7, 9.00 – 9.50, based on exercise sheets 1-4).
2 hours examination: 80%

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• Futures and Option Markets, where the derivative products are traded.

Example: European call option gives the holder the right (not obligation) to buy underlying asset at a prescribed time T for a specified price E.

Option market is massive! More money is invested in options than in the underlying securities. The main purpose of this course is to determine the price of options.

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Why stochastic differential equations (SDE's) and partial differential equations (PDE's)?

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• Continuous compounding:
In the limit $m \to \infty$, we obtain

$$V(T) = e^{rT} P \tag{3}$$

since $e = \lim_{z\to\infty} \left(1 + \frac{1}{z}\right)^{z}$. Throughout this course the interest rate r will be continuously compounded.

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• How to model the return?
Let us decompose the return into two parts: deterministic and stochastic

Modelling of Return

Return:

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{6}$$

where μ is a measure of the expected rate of growth of the stock price. In general, $\mu = \mu(S, t)$. In simpe models μ is taken to be constant ($\mu = 0.1 \div 0.3$).

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• σdW describes the stochastic change in the stock price, where dW stands for

$$\Delta W = W(t + \Delta t) - W(t)$$

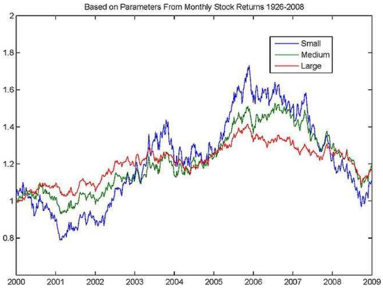
as $\Delta t \to 0$

• $W(t)$ is a Wiener process

• σ is the volatility ($\sigma = 0.2 \div 0.5$)

Stochastic differential equation for stock price

$$dS = \mu S dt + \sigma S dW \tag{7}$$



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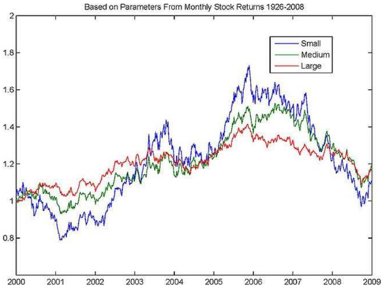

- Simple case: volatility $\sigma = 0$

$$dS = \mu S dt \tag{8}$$

Wiener process

Definition. The standard Wiener process $W(t)$ is a Gaussian process such that

- $W(t)$ has independent increments: if $u \leq v \leq s \leq t$, then $W(t) - W(s)$ and $W(v) - W(u)$ are independent

- $W(s+t) - W(s)$ is $N(0,t)$ and $W(0) = 0$

Clearly

- $\mathbb{E}W(t) = 0$ and $\mathbb{E}W^2 = t$, where $\mathbb{E}$ is the expectation operator.

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• The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X\,(\Delta t)^{\frac{1}{2}}$, where X is a random variable with normal distribution with zero mean and unit variance:

$$X \sim N\,(0,1)$$

• $\mathbb{E}\Delta W = 0$ and $\mathbb{E}(\Delta W)^2 = \Delta t$.